

Euclidean plane and its relatives

A minimalist introduction

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Introduction

This book is meant to be rigorous, conservative, elementary and minimalist. At the same time it includes about the maximum what students can absorb in one semester.

Approximately one-third of the material used to be covered in high school, but not any more.

The present book is based on the courses given by the author at the Pennsylvania State University as an introduction to the foundations of geometry. The lectures were oriented to sophomore and senior university students. These students already had a calculus course. In particular, they are familiar with the real numbers and continuity. It makes it possible to cover the material faster and in a more rigorous way than it could be done in high school.

Prerequisite

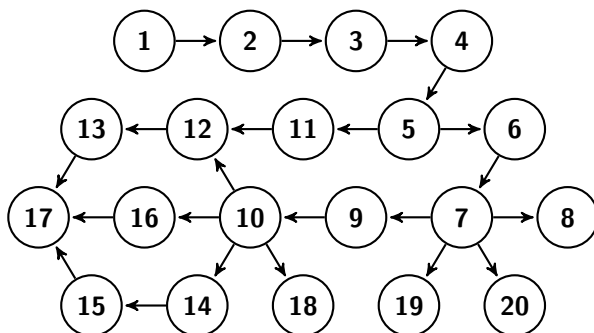
The students should be familiar with the following topics:

- ◇ Elementary set theory: $\in$, $\cup$, $\cap$, $\setminus$, $\subset$, $\times$.
- ◇ Real numbers: intervals, inequalities, algebraic identities.
- ◇ Limits, continuous functions and the intermediate value theorem.
- ◇ Standard functions: absolute value, natural logarithm, exponential function. Occasionally, trigonometric functions are used, but these parts can be ignored.
- ◇ Chapter 14 uses matrix algebra of 2×2 -matrices.
- ◇ To read Chapter 16, it is better to have some previous experience with the *scalar product*, also known as *dot product*.
- ◇ To read Chapter 18, it is better to have some previous experience with complex numbers.

Overview

We use the so called *metric approach* introduced by Birkhoff. It means that we define the Euclidean plane as a *metric space* which satisfies a list of properties (*axioms*). This way we minimize the tedious parts which are unavoidable in the more classical Hilbert's approach. At the same time the students have a chance to learn basic geometry of metric spaces.

Here is a dependency graph of the chapters.



In (1) we give all the definitions necessary to formulate the axioms; it includes metric space, lines, angle measure, continuous maps and congruent triangles.

Further we do Euclidean geometry: (2) Axioms and immediate corollaries; (3) Half-planes and continuity; (4) Congruent triangles; (5) Circles, motions, perpendicular lines; (6) Similar triangles and (7) Parallel lines — these are the first two chapters where we use Axiom V, an equivalent of Euclid's parallel postulate. In (8) we give the most classical theorem of triangle geometry; this chapter is included mainly as an illustration.

In the following two chapters we discuss geometry of circles on the Euclidean plane: (9) Inscribed angles; (10) Inversion. It will be used to construct the model of the hyperbolic plane.

Further we discuss non-Euclidean geometry: (11) Neutral geometry — geometry without the parallel postulate; (12) Conformal disc model — this is a construction of the hyperbolic plane, an example of a neutral plane which is not Euclidean. In (13) we discuss geometry of the constructed hyperbolic plane — this is the highest point in the book.

In the remaining chapters we discuss some additional topics: (14) Affine geometry; (15) Projective geometry; (16) Spherical geometry; (17) Projective model of the hyperbolic plane; (18) Complex coordinates; (19) Geometric constructions; (20) Area. The proofs in these chapters are not completely rigorous.

We encourage to use the visual assignments available at the author's

website.

Disclaimer

It is impossible to find the original reference to most of the theorems discussed here, so I do not even try to. Most of the proofs discussed in the book already appeared in the Euclid's Elements.

Recommended books

- ◊ Kiselev's textbook [10] — a classical book for school students; it should help if you have trouble following this book.
- ◊ Moise's book, [15] — should be good for further study.
- ◊ Greenberg's book [9] — a historical tour in the axiomatic systems of various geometries.
- ◊ Prasolov's book [16] is perfect to master your problem-solving skills.
- ◊ Akopyan's book [1] — a collection of problems formulated in figures.
- ◊ Methodologically my lectures were very close to Sharygin's textbook [18]. This is the greatest textbook in geometry for school students, I recommend it to anyone who can read Russian.

Acknowledgments

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Chapter 1

Preliminaries

What is the axiomatic approach?

In the axiomatic approach, one defines the plane as anything which satisfies a given list of properties. These properties are called *axioms*. The axiomatic system for the theory is like the rules for a game. Once the axiom system is fixed, a statement is considered to be true if it follows from the axioms and nothing else is considered to be true.

The formulations of the first axioms were not rigorous at all. For example, Euclid described a *line* as *breadthless length* and a *straight line* as a line which *lies evenly with the points on itself*. On the other hand, these formulations were sufficiently clear, so that one mathematician could understand the other.

The best way to understand an axiomatic system is to make one by yourself. Look around and choose a physical model of the Euclidean plane; imagine an infinite and perfect surface of a chalk board. Now try to collect the key observations about this model. Assume for now that we have intuitive understanding of such notions as *line* and *point*.

- (i) We can measure distances between points.
- (ii) We can draw a unique line which passes thru two given points.
- (iii) We can measure angles.
- (iv) If we rotate or shift we will not see the difference.
- (v) If we change scale we will not see the difference.

These observations are good to start with. Further we will develop the language to reformulate them rigorously.

What is a model?

The Euclidean plane can be defined rigorously the following way:

Define a point in the Euclidean plane as a pair of real numbers (x, y) and define the distance between the two points (x_1, y_1) and (x_2, y_2) by the following formula:

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

That is it! We gave a *numerical model* of Euclidean plane; it builds the Euclidean plane from the real numbers while the latter is assumed to be known.

Shortness is the main advantage of the model approach, but it is not intuitively clear why we define points and the distances this way.

On the other hand, the observations made in the previous section are intuitively obvious — this is the main advantage of the axiomatic approach.

Another advantage lies in the fact that the axiomatic approach is easily adjustable. For example, we may remove one axiom from the list, or exchange it to another axiom. We will do such modifications in Chapter 11 and further.

Metric spaces

The notion of metric space provides a rigorous way to say: “*we can measure distances between points*”. That is, instead of (i) on page 9, we can say “*Euclidean plane is a metric space*”.

1.1. Definition. *Let $\mathcal{X}$ be a nonempty set and d be a function which returns a real number $d(A, B)$ for any pair $A, B \in \mathcal{X}$. Then d is called metric on $\mathcal{X}$ if for any $A, B, C \in \mathcal{X}$, the following conditions are satisfied:*

(a) *Positiveness:*

$$d(A, B) \geq 0.$$

(b) *$A = B$ if and only if*

$$d(A, B) = 0.$$

(c) *Symmetry:*

$$d(A, B) = d(B, A).$$

(d) *Triangle inequality:*

$$d(A, C) \leq d(A, B) + d(B, C).$$

A metric space is a set with a metric on it. More formally, a metric space is a pair $(\mathcal{X}, d)$ where $\mathcal{X}$ is a set and d is a metric on $\mathcal{X}$.

The elements of $\mathcal{X}$ are called points of the metric space. Given two points $A, B \in \mathcal{X}$, the value $d(A, B)$ is called distance from A to B .

Examples

- ◇ *Discrete metric.* Let $\mathcal{X}$ be an arbitrary set. For any $A, B \in \mathcal{X}$, set $d(A, B) = 0$ if $A = B$ and $d(A, B) = 1$ otherwise. The metric d is called *discrete metric* on $\mathcal{X}$.
- ◇ *Real line.* Set of all real numbers ($\mathbb{R}$) with metric defined as

$$d(A, B) := |A - B|.$$

- ◇ *Metrics on the plane.* Let $\mathbb{R}^2$ denotes the set of all pairs (x, y) of real numbers. Assume $A = (x_A, y_A)$ and $B = (x_B, y_B)$. Consider the following metrics on $\mathbb{R}^2$:

- *Euclidean metric*, denoted by d_2 and defined as

$$d_2(A, B) = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2}.$$

- *Manhattan metric*, denoted by d_1 and defined as

$$d_1(A, B) = |x_A - x_B| + |y_A - y_B|.$$

- *Maximum metric*, denoted by d_∞ and defined as

$$d_\infty(A, B) = \max\{|x_A - x_B|, |y_A - y_B|\}.$$

1.2. Exercise. Prove that the following functions are metrics on $\mathbb{R}^2$: (a) d_1 ; (b) d_2 ; (c) d_∞ .

Shortcut for distance

Most of the time, we study only one metric on the space. Therefore, we will not need to name the metric function each time.

Given a metric space $\mathcal{X}$, the distance between points A and B will be further denoted by

$$AB \quad \text{or} \quad d_{\mathcal{X}}(A, B);$$

the latter is used only if we need to emphasize that A and B are points of the metric space $\mathcal{X}$.

For example, the triangle inequality can be written as

$$AC \leq AB + BC.$$

For the multiplication, we will always use “ $\cdot$ ”, so AB should not be confused with $A \cdot B$.

Isometries, motions and lines

In this section, we define *lines* in a metric space. Once it is done the sentence “We can draw a unique line which passes thru two given points.” becomes rigorous; see (ii) on page 9.

Recall that a map $f: \mathcal{X} \rightarrow \mathcal{Y}$ is a *bijection*, if it gives an exact pairing of the elements of two sets. Equivalently, $f: \mathcal{X} \rightarrow \mathcal{Y}$ is a bijection, if it has an *inverse*; that is, a map $g: \mathcal{Y} \rightarrow \mathcal{X}$ such that $g(f(A)) = A$ for any $A \in \mathcal{X}$ and $f(g(B)) = B$ for any $B \in \mathcal{Y}$.

1.3. Definition. Let $\mathcal{X}$ and $\mathcal{Y}$ be two metric spaces and $d_{\mathcal{X}}, d_{\mathcal{Y}}$ be their metrics. The map

$$f: \mathcal{X} \rightarrow \mathcal{Y}$$

is called *distance-preserving* if

$$d_{\mathcal{Y}}(f(A), f(B)) = d_{\mathcal{X}}(A, B)$$

for any $A, B \in \mathcal{X}$.

A bijective distance-preserving map is called an *isometry*.

Two metric spaces are called *isometric* if there exists an isometry from one to the other.

The isometry from a metric space to itself is also called a *motion* of the space.

1.4. Exercise. Show that any distance-preserving map is injective; that is, if $f: \mathcal{X} \rightarrow \mathcal{Y}$ is a distance-preserving map, then $f(A) \neq f(B)$ for any pair of distinct points $A, B \in \mathcal{X}$.

1.5. Exercise. Show that if $f: \mathbb{R} \rightarrow \mathbb{R}$ is a motion of the real line, then either (a) $f(x) = f(0) + x$ for any $x \in \mathbb{R}$, or (b) $f(x) = f(0) - x$ for any $x \in \mathbb{R}$.

1.6. Exercise. Prove that $(\mathbb{R}^2, d_1)$ is isometric to $(\mathbb{R}^2, d_{\infty})$.

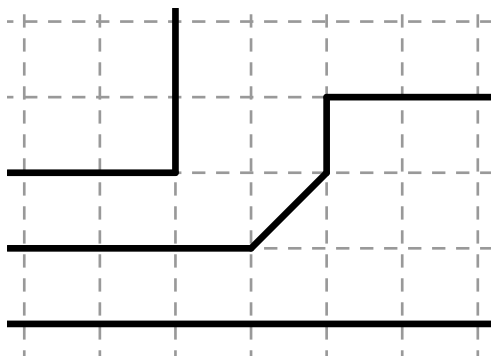
1.7. Advanced exercise. Describe all the motions of the Manhattan plane, defined on page 11.

If $\mathcal{X}$ is a metric space and $\mathcal{Y}$ is a subset of $\mathcal{X}$, then a metric on $\mathcal{Y}$ can be obtained by restricting the metric from $\mathcal{X}$. In other words, the distance between two points of $\mathcal{Y}$ is defined to be the distance between these points in $\mathcal{X}$. This way any subset of a metric space can be also considered as a metric space.

1.8. Definition. A subset ℓ of metric space is called a *line*, if it is isometric to the real line.

A triple of points that lie on one line is called *collinear*. Note that if A , B and C are collinear, $AC \geq AB$ and $AC \geq BC$, then $AC = AB + BC$.

Some metric spaces have no lines, for example discrete metrics. The picture shows examples of lines on the Manhattan plane $(\mathbb{R}^2, d_1)$.



1.9. Exercise. Consider the graph $y = |x|$ in $\mathbb{R}^2$. In which of the following spaces (a) $(\mathbb{R}^2, d_1)$, (b) $(\mathbb{R}^2, d_2)$, (c) $(\mathbb{R}^2, d_\infty)$ does it form a line? Why?

1.10. Exercise. How many points M are there on the line (AB) for which we have

- (a) $AM = MB$?
- (b) $AM = 2 \cdot MB$?

Half-lines and segments

Assume there is a line ℓ passing thru two distinct points P and Q . In this case we might denote ℓ as (PQ) . There might be more than one line thru P and Q , but if we write (PQ) we assume that we made a choice of such line.

Let $[PQ)$ denotes the *half-line* which starts at P and contains Q . Formally speaking, $[PQ)$ is a subset of (PQ) which corresponds to $[0, \infty)$ under an isometry $f: (PQ) \rightarrow \mathbb{R}$ such that $f(P) = 0$ and $f(Q) > 0$.

1.11. Exercise. Show that if $X \in [PQ)$, then $QX = |PX - PQ|$.

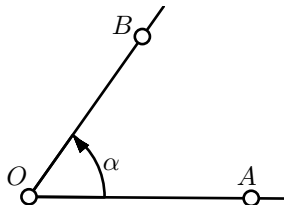
The subset of line (PQ) between P and Q is called the *segment* between P and Q and denoted by $[PQ]$. Formally, the segment can be defined as the intersection of two half-lines: $[PQ] = [PQ) \cap [QP)$.

Angles

Our next goal is to introduce *angles* and *angle measures*; after that, the statement “*we can measure angles*” will become rigorous; see (iii) on page 9.

An ordered pair of half-lines which start at the same point is called an *angle*. The angle AOB (also denoted by $\angle AOB$) is the pair of half-lines $[OA)$ and $[OB)$. In this case the point O is called the *vertex* of the angle.

Intuitively, the angle measure tells how much one has to rotate the first half-line counterclockwise, so it gets the position of the second half-line of the angle. The full turn is assumed to be $2\cdot\pi$; it corresponds to the angle measure in radians.¹



The angle measure of $\angle AOB$ is denoted by $\angle AOB$; it is a real number in the interval $(-\pi, \pi]$.

The notations $\angle AOB$ and $\angle A'O'B'$ look similar; they also have close but different meanings, which better not be confused. For example, the equality $\angle AOB = \angle A'O'B'$ means that $[OA) = [O'A')$ and $[OB) = [O'B')$; in particular, $O = O'$. On the other hand the equality $\angle AOB = \angle A'O'B'$ means only equality of two real numbers; in this case O may be distinct from O' .

Here is the first property of angle measure which will become a part of the axiom.

Given a half-line $[OA)$ and $\alpha \in (-\pi, \pi]$ there is a unique half-line $[OB)$ such that $\angle AOB = \alpha$.

Reals modulo $2\cdot\pi$

Consider three half-lines starting from the same point, $[OA)$, $[OB)$ and $[OC)$. They make three angles AOB , BOC and AOC , so the value $\angle AOC$ should coincide with the sum $\angle AOB + \angle BOC$ up to full rotation. This property will be expressed by the formula

$$\angle AOB + \angle BOC \equiv \angle AOC,$$

where “ $\equiv$ ” is a new notation which we are about to introduce. The last identity will become a part of the axiom.

¹For a while you may think that π is a positive real number that measures the size of a half turn in certain units. Its concrete value $\pi \approx 3.14$ will not be important for a long time.

We will write

$$\alpha \equiv \beta$$

or

$$\alpha \equiv \beta \pmod{2\cdot\pi}$$

if $\alpha = \beta + 2\cdot\pi\cdot n$ for some integer n . In this case we say

“ α is equal to β modulo $2\cdot\pi$ ”.

For example

$$-\pi \equiv \pi \equiv 3\cdot\pi \quad \text{and} \quad \frac{1}{2}\cdot\pi \equiv -\frac{3}{2}\cdot\pi.$$

The introduced relation “ $\equiv$ ” behaves as an equality sign, but

$$\dots \equiv \alpha - 2\cdot\pi \equiv \alpha \equiv \alpha + 2\cdot\pi \equiv \alpha + 4\cdot\pi \equiv \dots;$$

that is, if the angle measures differ by full turn, then they are considered to be the same.

With “ $\equiv$ ”, we can do addition, subtraction and multiplication with integer numbers without getting into trouble. That is, if

$$\alpha \equiv \beta \quad \text{and} \quad \alpha' \equiv \beta',$$

then

$$\alpha + \alpha' \equiv \beta + \beta', \quad \alpha - \alpha' \equiv \beta - \beta' \quad \text{and} \quad n\cdot\alpha \equiv n\cdot\beta$$

for any integer n . But “ $\equiv$ ” does not in general respect multiplication with non-integer numbers; for example

$$\pi \equiv -\pi \quad \text{but} \quad \frac{1}{2}\cdot\pi \not\equiv -\frac{1}{2}\cdot\pi.$$

1.12. Exercise. *Show that $2\cdot\alpha \equiv 0$ if and only if $\alpha \equiv 0$ or $\alpha \equiv \pi$.*

Continuity

The angle measure is also assumed to be continuous. Namely, the following property of angle measure will become a part of the axiom:

The function

$$\angle: (A, O, B) \mapsto \angle AOB$$

is continuous at any triple of points (A, O, B) such that $O \neq A$ and $O \neq B$ and $\angle AOB \neq \pi$.

To explain this property, we need to extend the notion of *continuity* to the functions between metric spaces. The definition is a straightforward generalization of the standard definition for the real-to-real functions.

Further, let $\mathcal{X}$ and $\mathcal{Y}$ be two metric spaces, and $d_{\mathcal{X}}$, $d_{\mathcal{Y}}$ be their metrics.

A map $f: \mathcal{X} \rightarrow \mathcal{Y}$ is called *continuous* at point $A \in \mathcal{X}$ if for any $\varepsilon > 0$ there is $\delta > 0$, such that

$$d_{\mathcal{X}}(A, A') < \delta \quad \Rightarrow \quad d_{\mathcal{Y}}(f(A), f(A')) < \varepsilon.$$

(Informally it means that sufficiently small changes of A result in arbitrarily small changes of $f(A)$.)

A map $f: \mathcal{X} \rightarrow \mathcal{Y}$ is called *continuous* if it is continuous at every point $A \in \mathcal{X}$.

One may define a continuous map of several variables the same way. Assume $f(A, B, C)$ is a function which returns a point in the space $\mathcal{Y}$ for a triple of points (A, B, C) in the space $\mathcal{X}$. The map f might be defined only for some triples in $\mathcal{X}$.

Assume $f(A, B, C)$ is defined. Then, we say that f is continuous at the triple (A, B, C) if for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$d_{\mathcal{Y}}(f(A, B, C), f(A', B', C')) < \varepsilon.$$

if $d_{\mathcal{X}}(A, A') < \delta$, $d_{\mathcal{X}}(B, B') < \delta$ and $d_{\mathcal{X}}(C, C') < \delta$.

1.13. Exercise. Let $\mathcal{X}$ be a metric space.

(a) Let $A \in \mathcal{X}$ be a fixed point. Show that the function

$$f(B) := d_{\mathcal{X}}(A, B)$$

is continuous at any point B .

(b) Show that $d_{\mathcal{X}}(A, B)$ is continuous at any pair $A, B \in \mathcal{X}$.

1.14. Exercise. Let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ be a metric spaces. Assume that the functions $f: \mathcal{X} \rightarrow \mathcal{Y}$ and $g: \mathcal{Y} \rightarrow \mathcal{Z}$ are continuous at any point, and $h = g \circ f$ is their composition; that is, $h(A) = g(f(A))$ for any $A \in \mathcal{X}$. Show that $h: \mathcal{X} \rightarrow \mathcal{Z}$ is continuous at any point.

1.15. Exercise. Show that any distance-preserving map is continuous at any point.

Congruent triangles

Our next goal is to give a rigorous meaning for (iv) on page 9. To do this, we introduce the notion of *congruent triangles* so instead of “if we rotate or shift we will not see the difference” we say that for triangles,

the side-angle-side congruence holds; that is, two triangles are congruent if they have two pairs of equal sides and the same angle measure between these sides.

An *ordered* triple of distinct points in a metric space $\mathcal{X}$, say A, B, C , is called a *triangle* ABC (briefly $\triangle ABC$). Note that the triangles ABC and ACB are considered as different.

Two triangles $A'B'C'$ and ABC are called *congruent* (written as $\triangle A'B'C' \cong \triangle ABC$) if there is a motion $f: \mathcal{X} \rightarrow \mathcal{X}$ such that

$$A' = f(A), \quad B' = f(B) \quad \text{and} \quad C' = f(C).$$

Let $\mathcal{X}$ be a metric space, and $f, g: \mathcal{X} \rightarrow \mathcal{X}$ be two motions. Note that the inverse $f^{-1}: \mathcal{X} \rightarrow \mathcal{X}$, as well as the composition $f \circ g: \mathcal{X} \rightarrow \mathcal{X}$ are also motions.

It follows that “ $\cong$ ” is an *equivalence relation*; that is, any triangle congruent to itself, and the following two conditions hold:

- ◇ If $\triangle A'B'C' \cong \triangle ABC$, then $\triangle ABC \cong \triangle A'B'C'$.
- ◇ If $\triangle A''B''C'' \cong \triangle A'B'C'$ and $\triangle A'B'C' \cong \triangle ABC$, then

$$\triangle A''B''C'' \cong \triangle ABC.$$

Note that if $\triangle A'B'C' \cong \triangle ABC$, then $AB = A'B'$, $BC = B'C'$ and $CA = C'A'$.

For a discrete metric, as well as some other metrics, the converse also holds. The following example shows that it does not hold in the Manhattan plane:

Example. Consider three points $A = (0, 1)$, $B = (1, 0)$ and $C = (-1, 0)$ on the Manhattan plane $(\mathbb{R}^2, d_1)$. Note that

$$d_1(A, B) = d_1(A, C) = d_1(B, C) = 2.$$

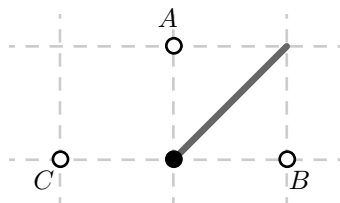
On one hand,

$$\triangle ABC \cong \triangle ACB.$$

Indeed, it follows since the map $(x, y) \mapsto (-x, y)$ is a motion of $(\mathbb{R}^2, d_1)$, which sends $A \mapsto A$, $B \mapsto C$ and $C \mapsto B$.

On the other hand,

$$\triangle ABC \not\cong \triangle BCA.$$



Indeed, arguing by contradiction, assume that $\triangle ABC \cong \triangle BCA$; that is, there is a motion f of $(\mathbb{R}^2, d_1)$ which sends $A \mapsto B$, $B \mapsto C$ and $C \mapsto A$.

We say that M is a midpoint of A and B if

$$d_1(A, M) = d_1(B, M) = \frac{1}{2} \cdot d_1(A, B).$$

Note that a point M is a midpoint of A and B if and only if $f(M)$ is a midpoint of B and C .

The set of midpoints for A and B is infinite, it contains all points (t, t) for $t \in [0, 1]$ (it is the dark gray segment on the picture above). On the other hand, the midpoint for B and C is unique (it is the black point on the picture). Thus, the map f cannot be bijective — a contradiction.

Chapter 2

Axioms

A system of axioms appears already in Euclid’s “Elements” — the most successful and influential textbook ever written.

The systematic study of geometries as axiomatic systems was triggered by the discovery of non-Euclidean geometry. The branch of mathematics, emerging this way, is called “Foundations of geometry”.

The most popular system of axiom was proposed in 1899 by David Hilbert. This is also the first rigorous system by modern standards. It contains twenty axioms in five groups, six “primitive notions” and three “primitive terms”; these are not defined in terms of previously defined concepts.

Later a number of different systems were proposed. It is worth mentioning the system of Alexandr Alexandrov [2] which is very intuitive and elementary, the system of Friedrich Bachmann [3] which is based on the concept of symmetry, and the system of Alfred Tarski [19], which was designed for analysis using mathematical logic.

We will use another system, which is very close to the one proposed by George Birkhoff in [6]. This system is based on the *key observations* (i)–(v) listed on page 9. The axioms use the notions of metric space, lines, angles, triangles, equalities modulo $2\cdot\pi$ ($\equiv$), the continuity of maps between metric spaces, and the congruence of triangles ($\cong$). All this discussed in the preliminaries.

Our system is build upon metric spaces. In particular, we use the real numbers as a building block. By that reason our approach is not purely axiomatic — we build the theory upon something else; it resembles a model-based introduction to Euclidean geometry discussed on page 10. We used this approach to minimize the tedious parts which are unavoidable in purely axiomatic foundations.

The axioms

- I. The *Euclidean plane* is a metric space with at least two points.
- II. There is one and only one line, that contains any two given distinct points P and Q in the Euclidean plane.
- III. Any angle AOB in the Euclidean plane defines a real number in the interval $(-\pi, \pi]$. This number is called *angle measure of $\angle AOB$* and denoted by $\angle AOB$. It satisfies the following conditions:
- (a) Given a half-line $[OA)$ and $\alpha \in (-\pi, \pi]$, there is a unique half-line $[OB)$, such that $\angle AOB = \alpha$.
- (b) For any points A, B and C , distinct from O we have

$$\angle AOB + \angle BOC \equiv \angle AOC.$$

- (c) The function

$$\angle: (A, O, B) \mapsto \angle AOB$$

is continuous at any triple of points (A, O, B) , such that $O \neq A$ and $O \neq B$ and $\angle AOB \neq \pi$.

- IV. In the Euclidean plane, we have $\triangle ABC \cong \triangle A'B'C'$ if and only if

$$A'B' = AB, \quad A'C' = AC, \quad \text{and} \quad \angle C'A'B' = \pm \angle CAB.$$

- V. If for two triangles $ABC, AB'C'$ in the Euclidean plane and for $k > 0$ we have

$$\begin{aligned} B' &\in [AB), & C' &\in [AC), \\ AB' &= k \cdot AB, & AC' &= k \cdot AC, \end{aligned}$$

then

$$B'C' = k \cdot BC, \quad \angle ABC = \angle AB'C', \quad \angle ACB = \angle AC'B'.$$

From now on, we can use no information about the Euclidean plane which does not follow from the five axioms above.

2.1. Exercise. Show that there are (a) an infinite set of points, (b) an infinite set of lines on the plane.

Lines and half-lines

2.2. Proposition.[✓] *Any two distinct lines intersect at most at one point.*

Proof. Assume that two lines ℓ and m intersect at two distinct points P and Q . Applying Axiom II, we get that $\ell = m$. $\square$

2.3. Exercise. *Suppose $A' \in [OA)$ and $A' \neq O$. Show that*

$$[OA) = [OA').$$

2.4. Proposition.[✓] *Given $r \geq 0$ and a half-line $[OA)$ there is a unique $A' \in [OA)$ such that $OA' = r$.*

Proof. According to definition of half-line, there is an isometry

$$f: [OA) \rightarrow [0, \infty),$$

such that $f(O) = 0$. By the definition of isometry, $OA' = f(A')$ for any $A' \in [OA)$. Thus, $OA' = r$ if and only if $f(A') = r$.

Since isometry has to be bijective, the statement follows. $\square$

Zero angle

2.5. Proposition.[✓] $\angle AOA = 0$ for any $A \neq O$.

Proof. According to Axiom IIIb,

$$\angle AOA + \angle AOA \equiv \angle AOA.$$

Subtract $\angle AOA$ from both sides, we get that $\angle AOA \equiv 0$.

By Axiom III, $-\pi < \angle AOA \leq \pi$; therefore $\angle AOA = 0$. $\square$

2.6. Exercise. *Assume $\angle AOB = 0$. Show that $[OA) = [OB)$.*

2.7. Proposition.[✓] *For any A and B distinct from O , we have*

$$\angle AOB \equiv -\angle BOA.$$

Proof. According to Axiom IIIb,

$$\angle AOB + \angle BOA \equiv \angle AOA$$

By Proposition 2.5, $\angle AOA = 0$. Hence the result. $\square$

[✓] A statement marked with “✓” if Axiom V was not used in its proof. Ignore this mark for a while; it will be important in Chapter 11, see page 80.

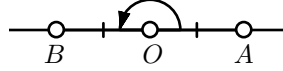
Straight angle

If $\angle AOB = \pi$, we say that $\angle AOB$ is a *straight angle*. Note that by Proposition 2.7, if $\angle AOB$ is a straight, then so is $\angle BOA$.

We say that point O lies *between* points A and B , if $O \neq A$, $O \neq B$ and $O \in [AB]$.

2.8. Theorem. ✓ *The angle AOB is straight if and only if O lies between A and B .*

Proof. By Proposition 2.4, we may assume that $OA = OB = 1$.



“If” part. Assume O lies between A and B . Set $\alpha = \angle AOB$.

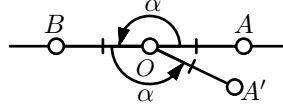
Applying Axiom IIIa, we get a half-line $[OA')$ such that $\alpha = \angle BOA'$. By Proposition 2.4, we can assume that $OA' = 1$. According to Axiom IV,

$$\triangle AOB \cong \triangle BOA'.$$

Let f denotes the corresponding motion of the plane; that is, f is a motion such that $f(A) = B$, $f(O) = O$ and $f(B) = A'$.

Then

$$(A'B) = f(AB) \ni f(O) = O.$$



Therefore, both lines (AB) and $(A'B)$ contain B and O . By Axiom II, $(AB) = (A'B)$.

By the definition of the line, (AB) contains exactly two points A and B on distance 1 from O . Since $OA' = 1$ and $A' \neq B$, we get that $A = A'$.

By Axiom IIIb and Proposition 2.5, we get that

$$\begin{aligned} 2 \cdot \alpha &= \angle AOB + \angle BOA' = \\ &= \angle AOB + \angle BOA \equiv \\ &\equiv \angle AOA = \\ &= 0 \end{aligned}$$

Therefore, by Exercise 1.12, α is either 0 or π .

Since $[OA] \neq [OB]$, we have that $\alpha \neq 0$, see Exercise 2.6. Therefore, $\alpha = \pi$.

“Only if” part. Suppose that $\angle AOB = \pi$. Consider the line (OA) and choose a point B' on (OA) so that O lies between A and B' .

From above, we have that $\angle AOB' = \pi$. Applying Axiom IIIa, we get that $[OB) = [OB')$. In particular, O lies between A and B . □

A triangle ABC is called *degenerate* if A , B and C lie on one line. The following corollary is just a reformulation of Theorem 2.8.

2.9. Corollary. *A triangle is degenerate if and only if one of its angles is equal to π or 0. Moreover in a degenerate triangle the angle measures are 0, 0 and π .*

2.10. Exercise. *Show that three distinct points A , O and B lie on one line if and only if*

$$2 \cdot \angle AOB \equiv 0.$$

2.11. Exercise. *Let A , B and C be three points distinct from O . Show that B , O and C lie on one line if and only if*

$$2 \cdot \angle AOB \equiv 2 \cdot \angle AOC.$$

2.12. Exercise. *Show that there is a nondegenerate triangle.*

Vertical angles

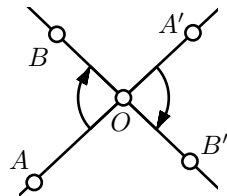
A pair of angles AOB and $A'OB'$ is called *vertical* if the point O lies between A and A' and between B and B' at the same time.

2.13. Proposition. *The vertical angles have equal measures.*

Proof. Assume that the angles AOB and $A'OB'$ are vertical. Note that $\angle AOA'$ and $\angle BOB'$ are straight. Therefore, $\angle AOA' = \angle BOB' = \pi$.

It follows that

$$\begin{aligned} 0 &= \angle AOA' - \angle BOB' \equiv \\ &\equiv \angle AOB + \angle BOA' - \angle BOA' - \angle A'OB' \equiv \\ &\equiv \angle AOB - \angle A'OB'. \end{aligned}$$



Since $-\pi < \angle AOB \leq \pi$ and $-\pi < \angle A'OB' \leq \pi$, we get that $\angle AOB = \angle A'OB'$. □

2.14. Exercise. *Assume O is the midpoint for both segments $[AB]$ and $[CD]$. Prove that $AC = BD$.*

Chapter 3

Half-planes

This chapter contains long proofs of intuitively evident statements. It is okay to skip it, but make sure you know definitions of positive/negative angles and that your intuition agrees with 3.7, 3.9, 3.10, 3.12 and 3.17.

Sign of an angle

The positive and negative angles can be visualized as *counterclockwise* and *clockwise* directions; formally, they are defined the following way:

- ◊ The angle AOB is called *positive* if $0 < \angle AOB < \pi$;
- ◊ The angle AOB is called *negative* if $\angle AOB < 0$.

Note that according to the above definitions the straight angle as well as the zero angle are neither positive nor negative.

3.1. Exercise. *Show that $\angle AOB$ is positive if and only if $\angle BOA$ is negative.*

3.2. Lemma. *Let $\angle AOB$ be straight. Then $\angle AOX$ is positive if and only if $\angle BOX$ is negative.*

Proof. Set $\alpha = \angle AOX$ and $\beta = \angle BOX$. Since $\angle AOB$ is straight,

$$\textcircled{1} \quad \alpha - \beta \equiv \pi.$$

It follows that $\alpha = \pi \Leftrightarrow \beta = 0$ and $\alpha = 0 \Leftrightarrow \beta = \pi$. In these two cases the sign of $\angle AOX$ and $\angle BOX$ are undefined.

In the remaining cases we have that $|\alpha| < \pi$ and $|\beta| < \pi$. If α and β have the same sign, then $|\alpha - \beta| < \pi$; the latter contradicts $\textcircled{1}$. Hence the statement follows. $\square$

3.3. Exercise. Assume that the angles AOB and BOC are positive. Show that

$$\angle AOB + \angle BOC + \angle COA = 2 \cdot \pi.$$

if $\angle COA$ is positive, and

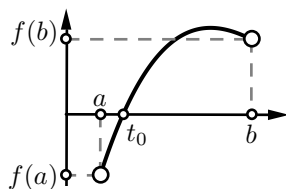
$$\angle AOB + \angle BOC + \angle COA = 0.$$

if $\angle COA$ is negative.

Intermediate value theorem

3.4. Intermediate value theorem. Let $f: [a, b] \rightarrow \mathbb{R}$ be a continuous function. Assume $f(a)$ and $f(b)$ have opposite signs, then $f(t_0) = 0$ for some $t_0 \in [a, b]$.

The intermediate value theorem is assumed to be known; it should be covered in any calculus course. We will use only the following corollary:



3.5. Corollary. Assume that for any $t \in [0, 1]$ we have three points in the plane O_t , A_t and B_t , such that

- (a) Each function $t \mapsto O_t$, $t \mapsto A_t$ and $t \mapsto B_t$ is continuous.
- (b) For any $t \in [0, 1]$, the points O_t , A_t and B_t do not lie on one line.

Then $\angle A_0 O_0 B_0$ and $\angle A_1 O_1 B_1$ have the same sign.

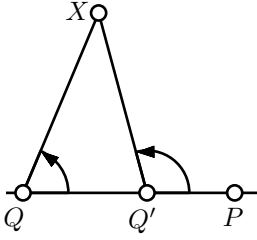
Proof. Consider the function $f(t) = \angle A_t O_t B_t$.

Since the points O_t , A_t and B_t do not lie on one line, Theorem 2.8 implies that $f(t) = \angle A_t O_t B_t \neq 0$ nor π for any $t \in [0, 1]$.

Therefore, by Axiom IIIc and Exercise 1.14, f is a continuous function. By the intermediate value theorem, $f(0)$ and $f(1)$ have the same sign; hence the result follows. $\square$

Same sign lemmas

3.6. Lemma. Assume $Q' \in [PQ)$ and $Q' \neq P$. Then for any $X \notin (PQ)$ the angles PQX and $PQ'X$ have the same sign.



Proof. By Proposition 2.4, for any $t \in [0, 1]$ there is a unique point $Q_t \in [PQ]$ such that

$$PQ_t = (1 - t) \cdot PQ + t \cdot PQ'.$$

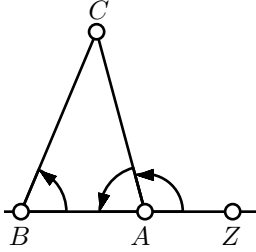
Note that the map $t \mapsto Q_t$ is continuous,

$$Q_0 = Q, \quad Q_1 = Q'$$

and for any $t \in [0, 1]$, we have that $P \neq Q_t$.

Applying Corollary 3.5, for $P_t = P$, Q_t and $X_t = X$, we get that $\angle PQX$ has the same sign as $\angle PQ'X$. $\square$

3.7. Signs of angles of a triangle. *In arbitrary nondegenerate triangle ABC , the angles ABC , BCA and CAB have the same sign.*



Proof. Choose a point $Z \in (AB)$ so that A lies between B and Z .

According to Lemma 3.6, the angles ZBC and ZAC have the same sign.

Note that $\angle ABC = \angle ZBC$ and

$$\angle ZAC + \angle CAB \equiv \pi.$$

Therefore, $\angle CAB$ has the same sign as $\angle ZAC$ which in turn has the same sign as $\angle ABC = \angle ZBC$.

Repeating the same argument for $\angle BCA$ and $\angle CAB$, we get the result. $\square$

3.8. Lemma. *Assume $[XY]$ does not intersect (PQ) , then the angles PQX and PQY have the same sign.*

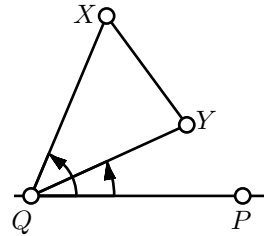
The proof is nearly identical to the one above.

Proof. According to Proposition 2.4, for any $t \in [0, 1]$ there is a point $X_t \in [XY]$, such that

$$XX_t = t \cdot XY.$$

Note that the map $t \mapsto X_t$ is continuous. Moreover, $X_0 = X$, $X_1 = Y$ and $X_t \notin (QP)$ for any $t \in [0, 1]$.

Applying Corollary 3.5, for $P_t = P$, $Q_t = Q$ and X_t , we get that $\angle PQX$ has the same sign as $\angle PQY$. $\square$



Half-planes

3.9. Proposition. Assume $X, Y \notin (PQ)$. Then the angles PQX and PQY have the same sign if and only if $[XY]$ does not intersect (PQ) .

Proof. The if-part follows from Lemma 3.8.

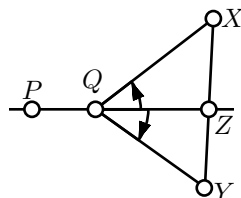
Assume $[XY]$ intersects (PQ) ; let Z denotes the point of intersection. Without loss of generality, we can assume $Z \neq P$.

Note that Z lies between X and Y . By Lemma 3.2, $\angle PZX$ and $\angle PZY$ have opposite signs. This proves the statement if $Z = Q$.

If $Z \neq Q$, then $\angle ZQX$ and $\angle QZX$ have opposite signs by 3.7. The same way we get that $\angle ZQY$ and $\angle QZY$ have opposite signs.

If Q lies between Z and P , then by Lemma 3.2 two pairs of angles $\angle PQX$, $\angle ZQX$ and $\angle PQY$, $\angle ZQY$ have opposite signs. It follows that $\angle PQX$ and $\angle PQY$ have opposite signs as required.

In the remaining case $[QZ] = [QP]$ and therefore $\angle PQX = \angle ZQX$ and $\angle PQY = \angle ZQY$. Hence again $\angle PQX$ and $\angle PQY$ have opposite signs as required. $\square$

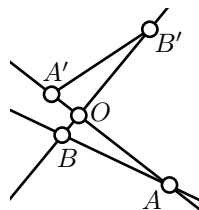


3.10. Corollary. The complement of a line (PQ) in the plane can be presented in a unique way as a union of two disjoint subsets called half-planes such that

- (a) Two points $X, Y \notin (PQ)$ lie in the same half-plane if and only if the angles PQX and PQY have the same sign.
- (b) Two points $X, Y \notin (PQ)$ lie in the same half-plane if and only if $[XY]$ does not intersect (PQ) .

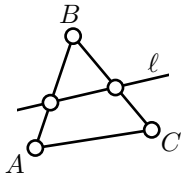
We say that X and Y lie on *one side* of (PQ) if they lie in one of the half-planes of (PQ) and we say that P and Q lie on the *opposite sides* of ℓ if they lie in the different half-planes of ℓ .

3.11. Exercise. Assume that the angles AOB and $A'OB'$ are vertical. Show that the line (AB) does not intersect the segment $[A'B']$.



Consider the triangle ABC . The segments $[AB]$, $[BC]$ and $[CA]$ are called *sides* of the triangle.

The following theorem follows from Corollary 3.10:



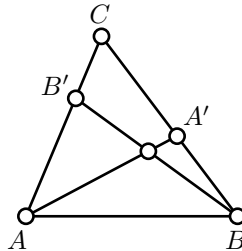
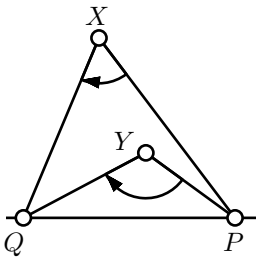
3.12. Pasch's theorem. Assume line ℓ does not pass thru any vertex a triangle. Then it intersects either two or zero sides of the triangle.

Proof. Assume that line ℓ intersects side $[AB]$ of the triangle ABC and does not pass thru A , B and C .

By Corollary 3.10, the vertexes A and B lie on opposite sides of ℓ .

The vertex C may lie on the same side with A and on opposite side with B or the other way around. By Corollary 3.10, in the first case, ℓ intersects side $[BC]$ and does not intersect $[AC]$; in the second case, ℓ intersects side $[AC]$ and does not intersect $[BC]$. Hence the statement follows. $\square$

3.13. Exercise. Show that two points $X, Y \notin (PQ)$ lie on the same side of (PQ) if and only if the angles PXQ and PYQ have the same sign.



3.14. Exercise. Let $\triangle ABC$ be a nondegenerate triangle, $A' \in [BC]$ and $B' \in [AC]$. Show that the segments $[AA']$ and $[BB']$ intersect.

3.15. Exercise. Assume that the points X and Y lie on opposite sides of the line (PQ) . Show that the half-line $[PX)$ does not intersect $[QY)$.

3.16. Advanced exercise. Note that the following quantity

$$\tilde{\angle} ABC = \begin{cases} \pi & \text{if } \angle ABC = \pi \\ -\angle ABC & \text{if } \angle ABC < \pi \end{cases}$$

can serve as the angle measure; that is, the axioms hold if one exchanges $\angle$ to $\tilde{\angle}$ everywhere.

Show that $\angle$ and $\tilde{\angle}$ are the only possible angle measures on the plane.

Show that without Axiom IIIc, this is no longer true.

Triangle with the given sides

Consider the triangle ABC . Set

$$a = BC, \quad b = CA, \quad c = AB.$$

Without loss of generality, we may assume that

$$a \leq b \leq c.$$

Then all three triangle inequalities for $\triangle ABC$ hold if and only if

$$c \leq a + b.$$

The following theorem states that this is the only restriction on a , b and c .

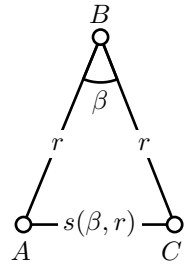
3.17. Theorem.✓ Assume that $0 < a \leq b \leq c \leq a + b$. Then there is a triangle with sides a , b and c ; that is, there is $\triangle ABC$ such that $a = BC$, $b = CA$ and $c = AB$.

The proof is given at the end of the section.

Assume $r > 0$ and $\pi > \beta > 0$. Consider the triangle ABC such that $AB = BC = r$ and $\angle ABC = \beta$. The existence of such a triangle follows from Axiom IIIa and Proposition 2.4.

Note that according to Axiom IV, the values β and r define the triangle ABC up to the congruence. In particular, the distance AC depends only on β and r . Set

$$s(\beta, r) := AC.$$



3.18. Proposition.✓ Given $r > 0$ and $\varepsilon > 0$, there is $\delta > 0$ such that if $0 < \beta < \delta$, then

$$s(r, \beta) < \varepsilon.$$

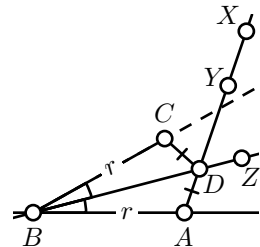
Proof. Fix two points A and B such that $AB = r$.

Choose a point X such that $\angle ABX$ is positive. Let $Y \in [AX)$ be the point such that $AY = \frac{\varepsilon}{8}$; it exists by Proposition 2.4.

Note that X and Y lie on the same side of (AB) ; therefore, $\angle ABY$ is positive. Set $\delta = \angle ABY$.

Assume $0 < \beta < \delta$, $\angle ABC = \beta$ and $BC = r$.

Applying Axiom IIIa, we can choose a half-line $[BZ)$ such that $\angle ABZ = \frac{1}{2}\beta$. Note that A



and Y lie on opposite sides of (BZ) . Therefore, (BZ) intersects $[AY]$; let D denotes the point of intersection.

Since $D \in (BZ)$, we get that $\angle ABD = \frac{\beta}{2}$ or $\frac{\beta}{2} - \pi$. The latter is impossible since D and Y lie on the same side of (AB) . Therefore,

$$\angle ABD = \angle DBC = \frac{\beta}{2}.$$

By Axiom IV, $\triangle ABD \cong \triangle CBD$. In particular,

$$\begin{aligned} AC &\leq AD + DC = \\ &= 2 \cdot AD \leq \\ &\leq 2 \cdot AY = \\ &= \frac{\varepsilon}{4} \end{aligned}$$

and hence the result. $\square$

3.19. Corollary. *Fix a real number $r > 0$ and two distinct points A and B . Then for any real number $\beta \in [0, \pi]$, there is a unique point C_β such that $BC_\beta = r$ and $\angle ABC_\beta = \beta$. Moreover, the map $\beta \mapsto C_\beta$ is a continuous map from $[0, \pi]$ to the plane.*

Proof. The existence and uniqueness of C_β follows from Axiom IIIa and Proposition 2.4.

Note that if $\beta_1 \neq \beta_2$, then

$$C_{\beta_1}C_{\beta_2} = s(r, |\beta_1 - \beta_2|).$$

Therefore, Proposition 3.18 implies that the map $\beta \mapsto C_\beta$ is continuous. $\square$

Proof of Theorem 3.17. Fix the points A and B such that $AB = c$. Given $\beta \in [0, \pi]$, let C_β denotes the point in the plane such that $BC_\beta = a$ and $\angle ABC = \beta$.

According to Corollary 3.19, the map $\beta \mapsto C_\beta$ is continuous. Therefore, the function $b(\beta) = AC_\beta$ is continuous (formally, it follows from Exercise 1.13 and Exercise 1.14).

Note that $b(0) = c - a$ and $b(\pi) = c + a$. Since $c - a \leq b \leq c + a$, by the intermediate value theorem (3.4) there is $\beta_0 \in [0, \pi]$ such that $b(\beta_0) = b$. Hence the result. $\square$

Given a positive real number r and a point O , the set Γ of all points on distance r from O is called a *circle* with *radius* r and *center* O .

3.20. Exercise. *Show that if two circles intersect if and only if*

$$|r_1 - r_2| \leq d \leq r_1 + r_2,$$

where d denotes the distances between their centers and r_1, r_2 — their radiuses.

Chapter 4

Congruent triangles

Side-angle-side condition

Our next goal is to give conditions which guarantee congruence of two triangles.

One of such conditions is given in Axiom IV; it states that if two pairs of sides of two triangles are equal, and the included angles are equal up to sign, then the triangles are congruent. This axiom is also called *side-angle-side congruence condition*, or briefly, *SAS congruence condition*.

Angle-side-angle condition

4.1. ASA condition.[✓] Assume that

$$AB = A'B', \quad \angle ABC = \pm \angle A'B'C', \quad \angle CAB = \pm \angle C'A'B'$$

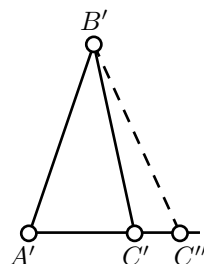
and $\triangle A'B'C'$ is nondegenerate. Then

$$\triangle ABC \cong \triangle A'B'C'.$$

Note that for degenerate triangles the statement does not hold. For example, consider one triangle with sides 1, 4, 5 and the other with sides 2, 3, 5.

Proof. According to Theorem 3.7, either

❶
$$\begin{aligned} \angle ABC &= \angle A'B'C', \\ \angle CAB &= \angle C'A'B' \end{aligned}$$



or

$$\begin{aligned} \textcircled{2} \quad \angle ABC &= -\angle A'B'C', \\ \angle CAB &= -\angle C'A'B'. \end{aligned}$$

Further we assume that $\textcircled{1}$ holds; the case $\textcircled{2}$ is analogous.

Let C'' be the point on the half-line $[A'C')$ such that $A'C'' = AC$.

By Axiom IV, $\triangle A'B'C'' \cong \triangle ABC$. Applying Axiom IV again, we get that

$$\angle A'B'C'' = \angle ABC = \angle A'B'C'.$$

By Axiom IIIa, $[B'C') = [BC'']$. Hence C'' lies on $(B'C')$ as well as on $(A'C')$.

Since $\triangle A'B'C'$ is not degenerate, $(A'C')$ is distinct from $(B'C')$. Applying Axiom II, we get that $C'' = C'$.

Therefore, $\triangle A'B'C' = \triangle A'B'C'' \cong \triangle ABC$. $\square$

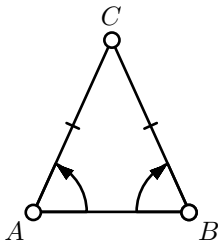
Isosceles triangles

A triangle with two equal sides is called *isosceles*; the remaining side is called the *base*.

4.2. Theorem. $\checkmark$ Assume $\triangle ABC$ is an isosceles triangle with the base $[AB]$. Then

$$\angle ABC \equiv -\angle BAC.$$

Moreover, the converse holds if $\triangle ABC$ is nondegenerate.



The following proof is due to Pappus of Alexandria.

Proof. Note that

$$CA = CB, \quad CB = CA, \quad \angle ACB \equiv -\angle BCA.$$

Therefore, by Axiom IV,

$$\triangle CAB \cong \triangle CBA.$$

Applying the theorem on the signs of angles of triangles (3.7) and Axiom IV again, we get that

$$\angle BAC \equiv -\angle ABC.$$

To prove the converse, we assume that $\angle CAB \equiv -\angle CBA$. By ASA condition 4.1, $\triangle CAB \cong \triangle CBA$. Therefore, $CA = CB$. $\square$

A triangle with three equal sides is called *equilateral*.

4.3. Exercise. Let $\triangle ABC$ be an equilateral triangle. Show that

$$\angle ABC = \angle BCA = \angle CAB.$$

Side-side-side condition

4.4. SSS condition. $\triangle ABC \cong \triangle A'B'C'$ if

$$A'B' = AB, \quad B'C' = BC \quad \text{and} \quad C'A' = CA.$$

Note that this condition is valid for degenerate triangles as well.

Proof. Choose C'' so that $A'C'' = A'C'$ and $\angle B'A'C'' = \angle BAC$. According to Axiom IV,

$$\triangle A'B'C'' \cong \triangle ABC.$$

It will suffice to prove that

$$\textcircled{3} \quad \triangle A'B'C' \cong \triangle A'B'C''.$$

The condition $\textcircled{3}$ trivially holds if $C'' = C'$. Thus, it remains to consider the case $C'' \neq C'$.

Clearly, the corresponding sides of $\triangle A'B'C'$ and $\triangle A'B'C''$ are equal. Hence the triangles $\triangle C'A'C''$ and $\triangle C'B'C''$ are isosceles. By Theorem 4.2, we have

$$\begin{aligned} \angle A'C''C' &\equiv -\angle A'C'C'', \\ \angle C'C''B' &\equiv -\angle C''C'B'. \end{aligned}$$

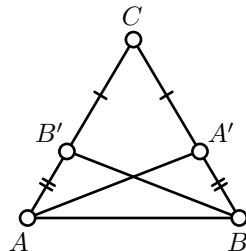
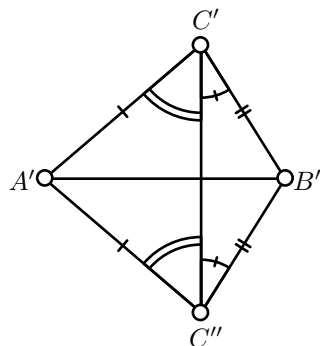
Adding them, we get that

$$\angle A'C''B' \equiv -\angle A'C'B'.$$

Applying Axiom IV again, we get $\textcircled{3}$. $\square$

4.5. Advanced exercise. Let M be the midpoint of the side $[AB]$ of $\triangle ABC$ and M' be the midpoint of the side $[A'B']$ of $\triangle A'B'C'$. Assume $C'A' = CA$, $C'B' = CB$ and $C'M' = CM$. Prove that

$$\triangle A'B'C' \cong \triangle ABC.$$



4.6. Exercise. Let $\triangle ABC$ be an isosceles triangle with the base $[AB]$. Suppose that the points $A' \in [BC]$ and $B' \in [AC]$ are such that $CA' = CB'$. Show that

- (a) $\triangle AA'C \cong \triangle BB'C$;
 (b) $\triangle ABB' \cong \triangle BAA'$.

4.7. Exercise. Show that if $AB + BC = AC$ then $B \in [AC]$.

4.8. Exercise. Let $\triangle ABC$ be a nondegenerate triangle and let f be a motion of the plane such that

$$f(A) = A, \quad f(B) = B \quad \text{and} \quad f(C) = C.$$

Show that f is the identity; that is, $f(X) = X$ for any point X on the plane.

On angle-side-side and side-angle-angle

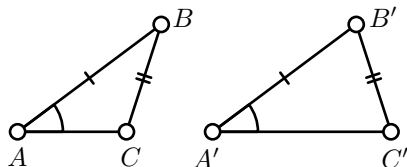
In each of the conditions SAS, ASA, and SSS we specify three corresponding parts of the triangles. Let us discuss other triples of corresponding parts.

The first triple is called *side-side-angle*, or briefly SSA; it specifies two sides and a non-included angle. This condition is not sufficient for congruence; that is, there are two nondegenerate triangles ABC and $A'B'C'$ such that

$$AB = A'B', \quad BC = B'C', \quad \angle BAC \equiv \pm \angle B'A'C',$$

but $\triangle ABC \not\cong \triangle A'B'C'$ and moreover $AC \neq A'C'$.

We will not use this negative statement in the sequel and therefore there is no need to prove it formally. An example can be guessed from the diagram.



The second triple is *side-angle-angle*,

or briefly SAA; it specifies one side and two angles one of which is opposite to the side. This provides a congruence condition; that is, if one of the triangles ABC and $A'B'C'$ is nondegenerate then

$$AB = A'B', \quad \angle ABC \equiv \pm \angle A'B'C', \quad \angle BCA \equiv \pm \angle B'C'A'$$

implies $\triangle ABC \cong \triangle A'B'C'$.

The SAA condition will not be used directly in the sequel. One proof of this condition can be obtained from ASA and the theorem on sum of angles of triangle proved below (see 7.10). For a more direct proof, see Exercise 11.6.

Another triple is called *angle-angle-angle*, or briefly AAA; by Axiom V, it is not sufficient for congruence. However AAA turns out to be a congruence condition in the hyperbolic plane; see 13.8.

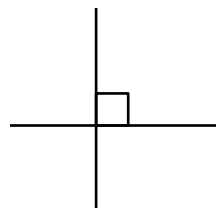
Chapter 5

Perpendicular lines

Right, acute and obtuse angles

- ◇ If $|\angle AOB| = \frac{\pi}{2}$, we say that $\angle AOB$ is *right*;
- ◇ If $|\angle AOB| < \frac{\pi}{2}$, we say that $\angle AOB$ is *acute*;
- ◇ If $|\angle AOB| > \frac{\pi}{2}$, we say that $\angle AOB$ is *obtuse*.

On the diagrams, the right angles will be marked with a little square, as shown.



If $\angle AOB$ is right, we say also that $[OA)$ is *perpendicular* to $[OB)$; it will be written as $[OA) \perp [OB)$.

From Theorem 2.8, it follows that two lines (OA) and (OB) are appropriately called *perpendicular*, if $[OA) \perp [OB)$. In this case we also write $(OA) \perp (OB)$.

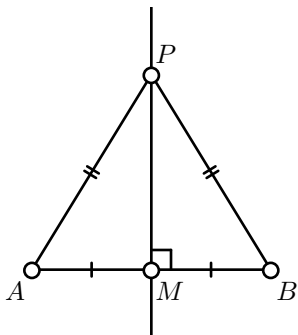
5.1. Exercise. Assume point O lies between A and B and $X \neq O$. Show that $\angle XO A$ is acute if and only if $\angle XO B$ is obtuse.

Perpendicular bisector

Assume M is the midpoint of the segment $[AB]$; that is, $M \in (AB)$ and $AM = MB$.

The line ℓ which passes thru M and perpendicular to (AB) , is called the *perpendicular bisector* to the segment $[AB]$.

5.2. Theorem.✓ Given distinct points A and B , all points equidistant from A and B and no others lie on the perpendicular bisector to $[AB]$.



Proof. Let M be the midpoint of $[AB]$.

Assume $PA = PB$ and $P \neq M$. According to SSS (4.4), $\triangle AMP \cong \triangle BMP$. Hence

$$\angle AMP = \pm \angle BMP.$$

Since $A \neq B$, we have “ $-$ ” in the above formula. Further,

$$\begin{aligned} \pi &= \angle AMB \equiv \\ &\equiv \angle AMP + \angle PMB \equiv \\ &\equiv 2 \cdot \angle AMP. \end{aligned}$$

That is, $\angle AMP = \pm \frac{\pi}{2}$. Therefore, P lies on the perpendicular bisector.

To prove the converse, suppose P is any point on the perpendicular bisector to $[AB]$ and $P \neq M$. Then $\angle AMP = \pm \frac{\pi}{2}$, $\angle BMP = \pm \frac{\pi}{2}$ and $AM = BM$. By SAS, $\triangle AMP \cong \triangle BMP$; in particular, $AP = BP$. $\square$

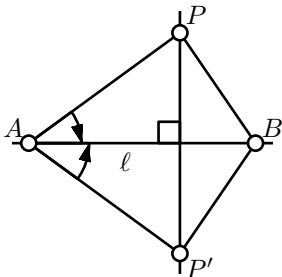
5.3. Exercise. Let ℓ be the perpendicular bisector to the segment $[AB]$ and X be an arbitrary point on the plane.

Show that $AX < BX$ if and only if X and A lie on the same side from ℓ .

5.4. Exercise. Let $\triangle ABC$ be nondegenerate. Show that $AC > BC$ if and only if $|\angle ABC| > |\angle CAB|$.

Uniqueness of a perpendicular

5.5. Theorem. $\checkmark$ There is one and only one line which passes thru a given point P and is perpendicular to a given line ℓ .



According to the above theorem, there is a unique point $Q \in \ell$ such that $(QP) \perp \ell$. This point Q is called the *foot point* of P on ℓ .

Proof. If $P \in \ell$, then both, existence and uniqueness, follow from Axiom III.

Existence for $P \notin \ell$. Let A and B be two distinct points of ℓ . Choose P' so that $AP' = AP$ and $\angle P'AB \equiv -\angle PAB$. According to Axiom IV, $\triangle AP'B \cong \triangle APB$. Therefore, $AP = AP'$ and $BP = BP'$.

According to Theorem 5.2, A and B lie on the perpendicular bisector to $[PP']$. In particular, $(PP') \perp (AB) = \ell$.

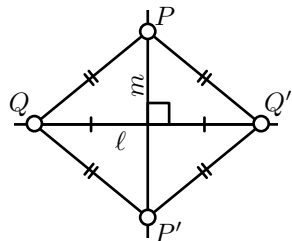
Uniqueness for $P \notin \ell$. From above we can choose a point P' in such a way that ℓ forms the perpendicular bisector to $[PP']$.

Assume $m \perp \ell$ and $m \ni P$. Then m is a perpendicular bisector to some segment $[QQ']$ of ℓ ; in particular, $PQ = PQ'$.

Since ℓ is the perpendicular bisector to $[PP']$, we get that $PQ = P'Q$ and $PQ' = P'Q'$. Therefore,

$$P'Q = PQ = PQ' = P'Q'.$$

By Theorem 5.2, P' lies on the perpendicular bisector to $[QQ']$, which is m . By Axiom II, $m = (PP')$. $\square$



Reflection

Assume the point P and the line (AB) are given. To find the *reflection* P' of P in (AB) , one drops a perpendicular from P onto (AB) , and continues it to the same distance on the other side.

According to Theorem 5.5, P' is uniquely determined by P .

Note that $P = P'$ if and only if $P \in (AB)$.

5.6. Proposition. Assume P' is a reflection of the point P in the line (AB) . Then $AP' = AP$ and if $A \neq P$, then $\angle BAP' \equiv -\angle BAP$.

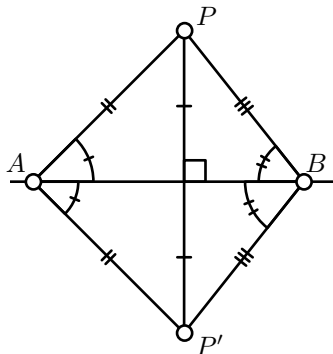
Proof. Note that if $P \in (AB)$, then $P = P'$. By Corollary 2.9, $\angle BAP = 0$ or π . Hence the statement follows.

If $P \notin (AB)$, then $P' \neq P$. By the construction of P' , the line (AB) is perpendicular bisector of $[PP']$. Therefore, according to Theorem 5.2, $AP' = AP$ and $BP' = BP$. In particular, $\triangle ABP' \cong \triangle ABP$. Therefore, $\angle BAP' = \pm \angle BAP$.

Since $P' \neq P$ and $AP' = AP$, we get that $\angle BAP' \neq \angle BAP$. That is, we are left with the case

$$\angle BAP' = -\angle BAP.$$

$\square$



5.7. Corollary. *The reflection in a line is a motion of the plane. Moreover, if $\triangle P'Q'R'$ is the reflection of $\triangle PQR$, then*

$$\angle Q'P'R' \equiv -\angle QPR.$$

Proof. From the construction, it follows that the composition of two reflections in the same line is the identity map. In particular, any reflection is a bijection.

Assume P' , Q' and R' denote the reflections of the points P , Q and R in (AB) . Let us show that

$$\textcircled{1} \quad P'Q' = PQ \quad \text{and} \quad \angle AP'Q' \equiv -\angle APQ.$$

Without loss of generality, we may assume that the points P and Q are distinct from A and B . By Proposition 5.6,

$$\begin{aligned} \angle BAP' &\equiv -\angle BAP, & \angle BAQ' &\equiv -\angle BAQ, \\ AP' &= AP, & AQ' &= AQ. \end{aligned}$$

It follows that $\angle P'AQ' \equiv -\angle PAQ$. Therefore $\triangle P'AQ' \cong \triangle PAQ$ and $\textcircled{1}$ follows.

Repeating the same argument for P and R , we get that

$$\angle AP'R' \equiv -\angle APR.$$

Subtracting the second identity in $\textcircled{1}$, we get that

$$\angle Q'P'R' \equiv -\angle QPR. \quad \square$$

5.8. Exercise. *Show that any motion of the plane can be presented as a composition of at most three reflections.*

Applying the exercise above and Corollary 5.7, we can divide the motions of the plane in two types, *direct* and *indirect motions*. The motion f is direct if

$$\angle Q'P'R' = \angle QPR$$

for any $\triangle PQR$ and $P' = f(P)$, $Q' = f(Q)$ and $R' = f(R)$; if instead we have

$$\angle Q'P'R' \equiv -\angle QPR$$

for any $\triangle PQR$, then the motion f is called indirect.

5.9. Exercise. *Let X and Y be the reflections of P in the lines (AB) and (BC) correspondingly. Show that*

$$\angle XBY \equiv 2 \cdot \angle ABC.$$

Perpendicular is shortest

5.10. Lemma.✓ Assume Q is the foot point of P on the line ℓ . Then the inequality

$$PX > PQ$$

holds for any point X on ℓ distinct from Q .

If P , Q and ℓ are as above, then PQ is called the *distance from P to ℓ* .

Proof. If $P \in \ell$, then the result follows since $PQ = 0$. Further we assume that $P \notin \ell$.

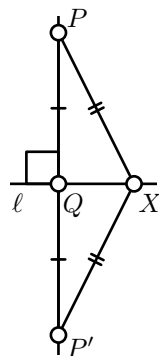
Let P' be the reflection of P in the line ℓ . Note that Q is the midpoint of $[PP']$ and ℓ is the perpendicular bisector of $[PP']$. Therefore

$$PX = P'X \quad \text{and} \quad PQ = P'Q = \frac{1}{2} \cdot PP'$$

Note that ℓ meets $[PP']$ only at the point Q . Therefore, $X \notin [PP']$; by the triangle inequality and Exercise 4.7,

$$PX + P'X > PP'$$

and hence the result: $PX > PQ$. $\square$



5.11. Exercise. Assume $\angle ABC$ is right or obtuse. Show that

$$AC > AB.$$

Circles

Recall that a circle with radius r and center O is the set of all points on distance r from O . We say that a point P lies *inside* of the circle if $OP < r$; if $OP > r$, we say that P lies *outside* of the circle.

5.12. Exercise. Let Γ be a circle and $P \notin \Gamma$. Assume a line ℓ is passing thru the point P and intersects Γ at two distinct points, X and Y . Show that P is inside Γ if and only if P lies between X and Y .

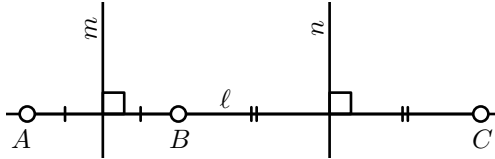
A segment between two points on a circle is called a *chord* of the circle. A chord passing thru the center of the circle is called its *diameter*.

5.13. Exercise. Assume two distinct circles Γ and Γ' have a common chord $[AB]$. Show that the line between centers of Γ and Γ' forms a perpendicular bisector to $[AB]$.

5.14. Lemma. *A line and a circle can have at most two points of intersection.*

Proof. Assume A , B and C are distinct points which lie on a line ℓ and a circle Γ with the center O .

Then $OA = OB = OC$; in particular, O lies on the perpendicular bisectors m and n to $[AB]$ and $[BC]$ correspondingly. Note that the



midpoints of $[AB]$ and $[BC]$ are distinct. Therefore, m and n are distinct. The latter contradicts the uniqueness of the perpendicular (Theorem 5.5). $\square$

5.15. Exercise. *Show that two distinct circles can have at most two points of intersection.*

In consequence of the above lemma, a line ℓ and a circle Γ might have 2, 1 or 0 points of intersections. In the first case the line is called *secant line*, in the second case it is *tangent line*; if P is the only point of intersection of ℓ and Γ , we say that ℓ is *tangent to Γ at P* .

Similarly, according Exercise 5.15, two circles might have 2, 1 or 0 points of intersections. If P is the only point of intersection of circles Γ and Γ' , we say that Γ is *tangent to Γ' at P* .

5.16. Lemma. *Let ℓ be a line and Γ be a circle with the center O . Assume P is a common point of ℓ and Γ . Then ℓ is tangent to Γ at P if and only if $(PO) \perp \ell$.*

Proof. Let Q be the foot point of O on ℓ .

Assume $P \neq Q$. Let P' denotes the reflection of P in (OQ) .

Note that $P' \in \ell$ and (OQ) is the perpendicular bisector of $[PP']$. Therefore, $OP = OP'$. Hence $P, P' \in \Gamma \cap \ell$; that is, ℓ is secant to Γ .

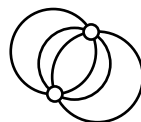
If $P = Q$, then according to Lemma 5.10, $OP < OX$ for any point $X \in \ell$ distinct from P . Hence P is the only point in the intersection $\Gamma \cap \ell$; that is, ℓ is tangent to Γ at P . $\square$

5.17. Exercise. *Let Γ and Γ' be two distinct circles with centers at O and O' correspondingly. Assume Γ meets Γ' at the point P . Show that Γ is tangent to Γ' if and only if O , O' and P lie on one line.*

5.18. Exercise. Let Γ and Γ' be two distinct circles with centers at O and O' and radii r and r' . Show that Γ is tangent to Γ' if and only if

$$OO' = r + r' \quad \text{or} \quad OO' = |r - r'|.$$

5.19. Exercise. Assume three circles intersect at two points as shown on the diagram. Prove that the centers of these circles lie on one line.



Geometric constructions

The *ruler-and-compass constructions* in the plane is the construction of points, lines, and circles using only an idealized ruler and compass. These construction problems provide a valuable source of exercises in geometry, which we will use further in the book. In addition, Chapter 19 is devoted completely to the subject.

The idealized ruler can be used only to draw a line thru the given two points. The idealized compass can be used only to draw a circle with a given center and radius. That is, given three points A , B and O we can draw the set of all points on distance AB from O . We may also mark new points in the plane as well as on the constructed lines, circles and their intersections (assuming that such points exist).

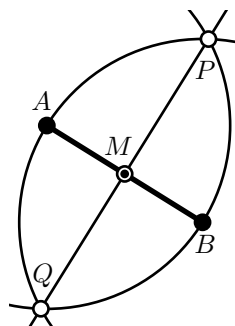
We can also look at the different set of construction tools. For example, we may only use the ruler or we may invent a new tool, say a tool which produces a midpoint for any given two points.

As an example, let us consider the following problem:

5.20. Construction of midpoint. Construct the midpoint of the given segment $[AB]$.

Construction.

1. Construct the circle with center at A which is passing thru B .
2. Construct the circle with center at B which is passing thru A .
3. Mark both points of intersection of these circles, label them with P and Q .
4. Draw the line (PQ) .
5. Mark the point of intersection of (PQ) and $[AB]$; this is the midpoint.



Typically, you need to prove that the construction produces what was expected. Here is a proof for the example above.

Proof. According to Theorem 5.2, (PQ) is the perpendicular bisector to $[AB]$. Therefore, $M = (AB) \cap (PQ)$ is the midpoint of $[AB]$. $\square$

5.21. Exercise. *Make a ruler-and-compass construction of a line thru a given point which is perpendicular to a given line.*

5.22. Exercise. *Make a ruler-and-compass construction of the center of a given circle.*

5.23. Exercise. *Make a ruler-and-compass construction of the lines tangent to a given circle which pass thru a given point.*

5.24. Exercise. *Given two circles Γ_1 and Γ_2 and a segment $[AB]$ make a ruler-and-compass construction of a circle with the radius AB , which is tangent to each circle Γ_1 and Γ_2 .*

Chapter 6

Similar triangles

Similar triangles

Two triangles $A'B'C'$ and ABC are called *similar* (briefly $\triangle A'B'C' \sim \triangle ABC$) if (1) their sides are proportional; that is,

$$\textcircled{1} \quad A'B' = k \cdot AB, \quad B'C' = k \cdot BC \quad \text{and} \quad C'A' = k \cdot CA$$

for some $k > 0$, and (2) the corresponding angles are equal up to sign:

$$\begin{aligned} \textcircled{2} \quad & \angle A'B'C' = \pm \angle ABC, \\ & \angle B'C'A' = \pm \angle BCA, \\ & \angle C'A'B' = \pm \angle CAB. \end{aligned}$$

Remarks.

- ◇ According to 3.7, in the above three equalities, the signs can be assumed to be the same.
- ◇ If $\triangle A'B'C' \sim \triangle ABC$ with $k = 1$ in $\textcircled{1}$, then $\triangle A'B'C' \cong \triangle ABC$.
- ◇ Note that “ $\sim$ ” is an *equivalence relation*. That is,
 - (i) $\triangle ABC \sim \triangle ABC$ for any $\triangle ABC$.
 - (ii) If $\triangle A'B'C' \sim \triangle ABC$, then

$$\triangle ABC \sim \triangle A'B'C'.$$

- (iii) If $\triangle A''B''C'' \sim \triangle A'B'C'$ and $\triangle A'B'C' \sim \triangle ABC$, then

$$\triangle A''B''C'' \sim \triangle ABC.$$

Using the new notation “ $\sim$ ”, we can reformulate Axiom V:

6.1. Reformulation of Axiom V. *If for the two triangles $\triangle ABC$, $\triangle AB'C'$, and $k > 0$ we have $B' \in [AB)$, $C' \in [AC)$, $AB' = k \cdot AB$ and $AC' = k \cdot AC$, then $\triangle ABC \sim \triangle AB'C'$.*

In other words, the Axiom V provides a condition which guarantees that two triangles are similar. Let us formulate three more such *similarity conditions*.

6.2. Similarity conditions. *Two triangles $\triangle ABC$ and $\triangle A'B'C'$ are similar if one of the following conditions holds:*

(SAS) *For some constant $k > 0$ we have*

$$AB = k \cdot A'B', \quad AC = k \cdot A'C'$$

$$\text{and } \angle BAC = \pm \angle B'A'C'.$$

(AA) *The triangle $A'B'C'$ is nondegenerate and*

$$\angle ABC = \pm \angle A'B'C', \quad \angle BAC = \pm \angle B'A'C'.$$

(SSS) *For some constant $k > 0$ we have*

$$AB = k \cdot A'B', \quad AC = k \cdot A'C', \quad CB = k \cdot C'B'.$$

Each of these conditions is proved by applying Axiom V with the SAS, ASA and SSS congruence conditions correspondingly (see Axiom IV and the conditions 4.1, 4.4).

Proof. Set $k = \frac{AB}{A'B'}$. Choose points $B'' \in [A'B')$ and $C'' \in [A'C')$, so that $A'B'' = k \cdot A'B'$ and $A'C'' = k \cdot A'C'$. By Axiom V, $\triangle A'B'C' \sim \triangle A'B''C''$.

Applying the SAS, ASA or SSS congruence condition, depending on the case, we get that $\triangle A'B''C'' \cong \triangle ABC$. Hence the result. $\square$

A bijection $X \leftrightarrow X'$ from a plane to itself is called *angle preserving transformation* if

$$\angle ABC = \angle A'B'C'$$

for any triangle ABC and its image $\triangle A'B'C'$.

6.3. Exercise. *Show that any angle-preserving transformation of the plane multiplies all the distance by a fixed constant.*

Pythagorean theorem

A triangle is called *right* if one of its angles is right. The side opposite the right angle is called the *hypotenuse*. The sides adjacent to the right angle are called *legs*.

6.4. Theorem. Assume $\triangle ABC$ is a right triangle with the right angle at C . Then

$$AC^2 + BC^2 = AB^2.$$

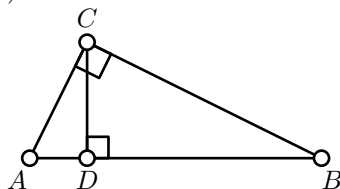
Proof. Let D be the foot point of C on (AB) .

According to Lemma 5.10,

$$AD < AC < AB$$

and

$$BD < BC < AB.$$



Therefore, D lies between A and B ; in particular,

$$\textcircled{3} \quad AD + BD = AB.$$

Note that by the AA similarity condition, we have

$$\triangle ADC \sim \triangle ACB \sim \triangle CDB.$$

In particular,

$$\textcircled{4} \quad \frac{AD}{AC} = \frac{AC}{AB} \quad \text{and} \quad \frac{BD}{BC} = \frac{BC}{BA}.$$

Let us rewrite the two identities in $\textcircled{4}$:

$$AC^2 = AB \cdot AD \quad \text{and} \quad BC^2 = AB \cdot BD.$$

Summing up these two identities and applying $\textcircled{3}$, we get that

$$AC^2 + BC^2 = AB \cdot (AD + BD) = AB^2. \quad \square$$

The idea in the proof above appears in the Elements, see [8, X.33], but the proof given there [8, I.47] is different; it uses area method discussed in Chapter 20.

6.5. Exercise. Assume A, B, C and D are as in the proof above. Show that

$$CD^2 = AD \cdot BD.$$

The following exercise is the converse to the Pythagorean theorem.

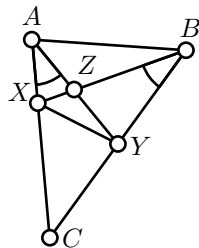
6.6. Exercise. Assume that ABC is a triangle such that

$$AC^2 + BC^2 = AB^2.$$

Prove that the angle at C is right.

Method of similar triangles

The proof of the Pythagorean theorem given above is a good example using the *method of similar triangles*. To apply this method, one has to search for pairs of similar triangles and then use the proportionality of corresponding sides and/or equalities of corresponding angles. Finding such pairs might be tricky at first.



6.7. Exercise. Let ABC be a nondegenerate triangle and the points X , Y and Z as on the diagram. Assume $\angle CAZ \equiv \angle XBC$. Find four pairs of similar triangles with these six points as the vertices and prove their similarity.

6.1 Ptolemy's inequality

A *quadrilateral* is defined as an ordered quadruple of distinct points in the plane. These 4 points are called *vertices* of quadrilateral. The quadrilateral $ABCD$ will be also denoted by $\square ABCD$.

Given a quadrilateral $ABCD$, the four segments $[AB]$, $[BC]$, $[CD]$ and $[DA]$ are called *sides* of $\square ABCD$; the remaining two segments $[AC]$ and $[BD]$ are called *diagonals* of $\square ABCD$.

6.8. Ptolemy's inequality. In any quadrilateral, the product of diagonals can not exceed the sum of the products of its opposite sides; that is,

$$AC \cdot BD \leq AB \cdot CD + BC \cdot DA$$

for any $\square ABCD$.

We will present a classical proof of this inequality using the method of similar triangles with an additional construction. This proof is given as an illustration — it will not be used further in the sequel.

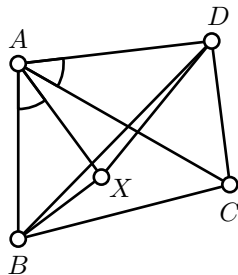
Proof. Consider the half-line $[AX)$ such that $\angle BAX = \angle CAD$. In this case $\angle XAD = \angle BAC$ since adding $\angle BAX$ or $\angle CAD$ to the corresponding sides produces $\angle BAD$. Note that we can choose X so that

$$AX = \frac{AB}{AC} \cdot AD.$$

In this case we have

$$\frac{AX}{AD} = \frac{AB}{AC},$$

$$\frac{AX}{BA} = \frac{AD}{AC}.$$



Hence

$$\triangle BAX \sim \triangle CAD, \quad \triangle XAD \sim \triangle BAC.$$

Therefore

$$\frac{BX}{CD} = \frac{AB}{AC}, \quad \frac{XD}{BC} = \frac{AD}{AC},$$

or, equivalently

$$AC \cdot BX = AB \cdot CD, \quad AC \cdot XD = BC \cdot AD.$$

Adding these two equalities we get

$$AC \cdot (BX + XD) = AB \cdot CD + BC \cdot AD.$$

It remains to apply the triangle inequality, $BD \leq BX + XD$. □

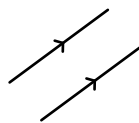
Using the proof above together with Theorem 9.10, one can show that the equality holds only if the vertexes A , B , C and D appear on a line or a circle in the same cyclic order; see also 10.12 for another proof of the equality case. Exercise 18.2 below suggests another proof of Ptolemy's inequality using complex coordinates.

Chapter 7

Parallel lines

Parallel lines

In consequence of Axiom II, any two distinct lines ℓ and m have either one point in common or none. In the first case they are *intersecting* (briefly $\ell \nparallel m$); in the second case, ℓ and m are said to be *parallel* (briefly $\ell \parallel m$); in addition, a line is always regarded as parallel to itself.



To emphasize that two lines on a diagram are parallel we will mark them with arrows of the same type.

7.1. Proposition.✓ *Let ℓ , m and n be three lines. Assume that $n \perp m$ and $m \perp \ell$. Then $\ell \parallel n$.*

Proof. Assume the contrary; that is, $\ell \nparallel n$. Then there is a point, say Z , of intersection of ℓ and n . Then by Theorem 5.5, $\ell = n$.

Since any line is parallel to itself, we have that $\ell \parallel n$ — a contradiction. $\square$

7.2. Theorem. *For any point P and any line ℓ there is a unique line m which passes thru P and is parallel to ℓ .*

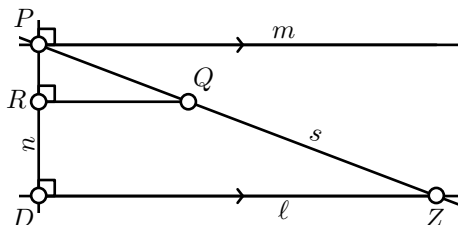
The above theorem has two parts, existence and uniqueness. In the proof of uniqueness we will use the method of similar triangles.

Proof; existence. Apply Theorem 5.5 two times, first to construct the line m thru P which is perpendicular to ℓ , and second to construct the line n thru P which is perpendicular to m . Then apply Proposition 7.1.

Uniqueness. If $P \in \ell$, then $m = \ell$ by the definition of parallel lines. Further we assume $P \notin \ell$.

Let us construct the lines $n \ni P$ and $m \ni P$ as in the proof of existence, so $m \parallel \ell$.

Assume there is yet another line $s \ni P$ which is distinct from m and parallel to ℓ . Choose a point $Q \in s$ which lies with ℓ on the same side from m . Let R be the foot point of Q on n .



Let D be the point of intersection of n and ℓ . According to Proposition 7.1 $(QR) \parallel m$. Therefore, Q , R and ℓ lie on the same side of m . In particular, $R \in [PD)$.

Choose $Z \in [PQ)$ such that

$$\frac{PZ}{PQ} = \frac{PD}{PR}.$$

By SAS similarity condition (or equivalently by Axiom V) we have that $\triangle RPQ \sim \triangle DPZ$; therefore $(ZD) \perp (PD)$. It follows that Z lies on ℓ and s — a contradiction. $\square$

7.3. Corollary. Assume ℓ , m and n are lines such that $\ell \parallel m$ and $m \parallel n$. Then $\ell \parallel n$.

Proof. Assume the contrary; that is, $\ell \not\parallel n$. Then there is a point $P \in \ell \cap n$. By Theorem 7.2, $n = \ell$ — a contradiction. $\square$

Note that from the definition, we have that $\ell \parallel m$ if and only if $m \parallel \ell$. Therefore, according to the above corollary, “ $\parallel$ ” is an *equivalence relation*. That is, for any lines ℓ , m and n the following conditions hold:

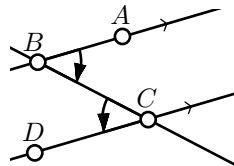
- (i) $\ell \parallel \ell$;
- (ii) if $\ell \parallel m$, then $m \parallel \ell$;
- (iii) if $\ell \parallel m$ and $m \parallel n$, then $\ell \parallel n$.

7.4. Exercise. Let k , ℓ , m and n be lines such that $k \perp \ell$, $\ell \perp m$ and $m \perp n$. Show that $k \not\parallel n$.

7.5. Exercise. Make a ruler-and-compass construction of a line thru a given point which is parallel to a given line.

Transversal property

If the line t intersects each line ℓ and m at one point, then we say that t is a *transversal* to ℓ and m . For example, on the diagram, line (CB) is a transversal to (AB) and (CD) .



7.6. Transversal property. $(AB) \parallel (CD)$ if and only if

$$\textcircled{1} \quad 2 \cdot (\angle ABC + \angle BCD) \equiv 0.$$

Equivalently

$$\angle ABC + \angle BCD \equiv 0 \quad \text{or} \quad \angle ABC + \angle BCD \equiv \pi.$$

Moreover, if $(AB) \neq (CD)$, then in the first case A and D lie on the opposite sides of (BC) , in the second case A and D lie on the same sides of (BC) .

The following lemma will be used several times in the sequel.

7.7. Lemma. Let ABC be a nondegenerate triangle. Then there is a point $X \notin (AC)$ such that

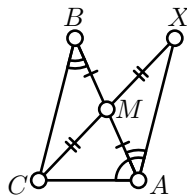
$$\angle BAX = \angle ABC$$

and, moreover, X lies with B on the same side of (AC) .

Proof. Let M be the midpoint of $[AB]$. Choose $X \in (CM)$ distinct from C so that $XM = CM$.

Note that the angles CMB and XMA are vertical; in particular,

$$\textcircled{2} \quad \angle CMB = \angle XMA.$$



By construction, $AM = BM$ and $CM = XM$. Therefore,

$$\triangle CMB \cong \triangle XMA;$$

in particular $\angle ABC \equiv \pm \angle BAX$. Since angles of $\triangle AMX$ have the same sign (3.7), by $\textcircled{2}$ we can not have a minus sign in the last equality; therefore

$$\angle ABC \equiv \angle BAX.$$

By construction, the line segments $[BM]$ and $[MX]$ do not cross the line (AC) . Therefore the points X and B lie on the same side of (AC) (see 3.10); in particular, $X \notin (AC)$. $\square$

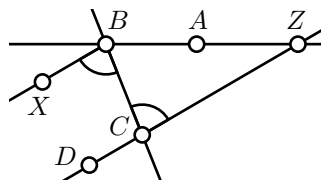
Proof of transversal property; “only-if” part. We need to show that if $(AB) \nparallel (CD)$, then ❶ does not hold.

Since $(AB) \nparallel (CD)$, there is a unique point $Z \in (AB) \cap (CD)$. Since Z lies on (AB) and on (CD) , Exercise 2.11 implies that

$$2 \cdot \angle ABC \equiv 2 \cdot \angle ZBC, \quad 2 \cdot \angle BCD \equiv 2 \cdot \angle BCZ.$$

Note that the triangle ZBC is nondegenerate. By Lemma 7.7, there is a point $X \notin (ZB)$ such that $\angle CBX = \angle BCZ$. Applying Exercise 2.10 and the equalities above, we get that

$$\begin{aligned} 0 \neq 2 \cdot \angle ZBX &\equiv \\ &\equiv 2 \cdot (\angle ZBC + \angle CBX) \\ &\equiv 2 \cdot (\angle ZBC + \angle BCZ) \\ &\equiv 2 \cdot (\angle ABC + \angle BCD); \end{aligned}$$



that is, ❶ does not hold.

“If”-part. Given points A, B and C there is unique line (CD) such that ❶ holds. Indeed, suppose there are two such lines (CD) and (CD') , then

$$2 \cdot (\angle ABC + \angle BCD) \equiv 2 \cdot (\angle ABC + \angle BCD') \equiv 0.$$

Therefore

$$2 \cdot \angle BCD \equiv 2 \cdot \angle BCD'.$$

It follows that $2 \cdot \angle DCD' \equiv 0$; that is, $D' \in (CD)$, or equivalently the line (CD) coincides with the line (CD') .

From the “only-if” part we know that if ❶ holds then $(CD) \parallel (AB)$. By Theorem 7.2 there is unique line thru C that is parallel to (AB) . Therefore it must be the same line (CD) such that ❶ hold. $\square$

7.8. Exercise. Let $\triangle ABC$ be a nondegenerate triangle. Assume B' and C' are points on sides $[AB]$ and $[AC]$ such that $(B'C') \parallel (BC)$. Show that $\triangle ABC \sim \triangle AB'C'$.

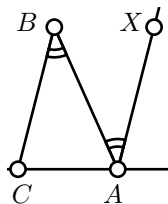
7.9. Exercise. Trisect a given segment with a ruler and a compass.

Angles of triangles

7.10. Theorem. In any $\triangle ABC$, we have

$$\angle ABC + \angle BCA + \angle CAB \equiv \pi.$$

Proof. First note that if $\triangle ABC$ is degenerate, then the equality follows from Corollary 2.9. Further we assume that $\triangle ABC$ is nondegenerate.



By Lemma 7.7 there is a point X that lies with B on the same side of (AC) such that $\angle BAX = \angle ABC$. In particular

$$2 \cdot (\angle XAB + \angle ABC) \equiv 0.$$

By transversal property, we get that $(AX) \parallel (CB)$.

Now, let us apply the transversal property (7.6) to the parallel lines (BC) and (AX) and their transversal (CA) . Since B and X lie on the same side of (CA) , we have

$$\textcircled{3} \quad \angle BCA + \angle CAX \equiv \pi.$$

Since $\angle BAX = \angle ABC$, we have

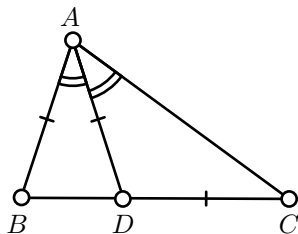
$$\angle CAX \equiv \angle CAB + \angle ABC$$

The latter identity and $\textcircled{3}$ imply the theorem. $\square$

7.11. Exercise. Let $\triangle ABC$ be a nondegenerate triangle. Assume there is a point $D \in [BC]$ such that

$$\angle BAD \equiv \angle DAC, \quad BA = AD = DC.$$

Find the angles of $\triangle ABC$.

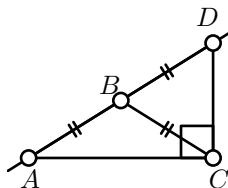


7.12. Exercise. Show that

$$|\angle ABC| + |\angle BCA| + |\angle CAB| = \pi$$

for any $\triangle ABC$.

7.13. Exercise. Let $\triangle ABC$ be an isosceles nondegenerate triangle with the base $[AC]$. Consider the point D on the extension of the side $[AB]$ such that $AB = BD$. Show that $\angle ACD$ is right.



7.14. Exercise. Let $\triangle ABC$ be an isosceles nondegenerate triangle with base $[AC]$. Assume that a circle is passing thru A , centered at a point on $[AB]$ and tangent to (BC) at the point X . Show that $\angle CAX = \pm \frac{\pi}{4}$.

7.15. Exercise. Show that for any quadrilateral $ABCD$, we have

$$\angle ABC + \angle BCD + \angle CDA + \angle DAB \equiv 0.$$

Parallelograms

A quadrilateral $ABCD$ in the Euclidean plane is called *nondegenerate* if any three points from A, B, C, D do not lie on one line.

A nondegenerate quadrilateral is called a *parallelogram* if its opposite sides are parallel.

7.16. Lemma. *If $\square ABCD$ is a parallelogram, then*

$$(a) \angle DAB = \angle BCD;$$

$$(b) AB = CD.$$

Proof. Since $(AB) \parallel (CD)$, the points C and D lie on the same side of (AB) . Hence $\angle ABD$ and $\angle ABC$ have the same sign.

The same way we get that $\angle CBD$ and $\angle CBA$ have the same sign.

Since $\angle ABC \equiv -\angle CBA$, we get that the angles DBA and DBC have opposite signs; that is, A and C lie on opposite sides of (BD) .

According to the transversal property (7.6),

$$\angle BDC \equiv -\angle DBA \quad \text{and} \quad \angle DBC \equiv -\angle BDA.$$

By the ASA condition $\triangle ABD \cong \triangle CDB$. The latter implies both statements in the lemma. $\square$

7.17. Exercise. *Assume $ABCD$ is a quadrilateral such that*

$$AB = CD = BC = DA.$$

Show that $ABCD$ is a parallelogram.

A quadrilateral as in the exercise above is called a *rhombus*.

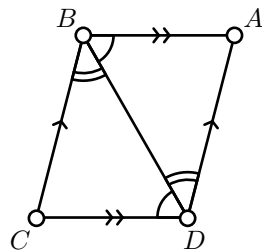
7.18. Exercise. *Show that diagonals of a parallelogram intersect each other at their midpoints.*

A quadrilateral $ABCD$ is called a *rectangle* if the angles ABC , BCD , CDA and DAB are right. Note that according to the transversal property 7.6, any rectangle is a parallelogram.

A rectangle with equal sides is called a *square*.

7.19. Exercise. *Show that the parallelogram $ABCD$ is a rectangle if and only if $AC = BD$.*

7.20. Exercise. *Show that the parallelogram $ABCD$ is a rhombus if and only if $(AC) \perp (BD)$.*



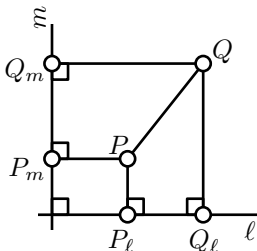
Assume $\ell \parallel m$, and $X, Y \in m$. Let X' and Y' denote the foot points of X and Y on ℓ . Note that $\square XYY'X'$ is a rectangle. By Lemma 7.16, $XX' = YY'$. That is, any point on m lies on the same distance from ℓ . This distance is called the *distance between ℓ and m* .

Method of coordinates

The following exercise is important; it shows that our axiomatic definition agrees with the model definition described on page 11.

7.21. Exercise. Let ℓ and m be perpendicular lines in the Euclidean plane. Given a point P , let P_ℓ and P_m denote the foot points of P on ℓ and m correspondingly.

- Show that for any $X \in \ell$ and $Y \in m$ there is a unique point P such that $P_\ell = X$ and $P_m = Y$.
- Show that $PQ^2 = P_\ell Q_\ell^2 + P_m Q_m^2$ for any pair of points P and Q .
- Conclude that the plane is isometric to $(\mathbb{R}^2, d_2)$; see page 11.



Once this exercise is solved, we can apply the method of coordinates to solve any problem in Euclidean plane geometry. This method is powerful, but it is often considered as a bad style.

7.22. Exercise. Use the Exercise 7.21 to give an alternative proof of Theorem 3.17 in the Euclidean plane.

That is, prove that given the real numbers a , b and c such that

$$0 < a \leq b \leq c \leq a + b,$$

there is a triangle ABC such that $a = BC$, $b = CA$ and $c = AB$.

7.23. Exercise. Consider two distinct points $A = (x_A, y_A)$ and $B = (x_B, y_B)$ on the coordinate plane. Show that the perpendicular bisector to $[AB]$ is described by the equation

$$2 \cdot (x_B - x_A) \cdot x + 2 \cdot (y_B - y_A) \cdot y = x_B^2 + y_B^2 - x_A^2 - y_A^2.$$

Conclude that any line is described by an equation

$$a \cdot x + b \cdot y = c$$

for some constants a , b and c such that $a \neq 0$ or $b \neq 0$.

7.24. Exercise. Show that for fixed real values a , b and c the equation

$$x^2 + y^2 + a \cdot x + b \cdot y + c = 0$$

describes a circle, one-point set or empty set.

Show that if it is a circle then it has center $(-\frac{a}{2}, -\frac{b}{2})$ and the radius $r = \frac{1}{2} \cdot \sqrt{a^2 + b^2 - 4 \cdot c}$.

7.25. Exercise. *Use the previous exercise to show that given two distinct point A and B and positive real number $k \neq 1$, the locus of points M such that $AM = k \cdot BM$ is a circle.*

The circle in the exercise above is an example of the so called *Apollo-nian circle*.

Chapter 8

Triangle geometry

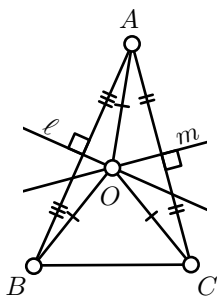
Triangle geometry is the study of the properties of triangles, including associated centers and circles.

We discuss the most basic results in triangle geometry, mostly to show that we have developed sufficient machinery to prove things.

Circumcircle and circumcenter

8.1. Theorem. *Perpendicular bisectors to the sides of any nondegenerate triangle intersect at one point.*

The point of intersection of the perpendicular bisectors is called *circumcenter*. It is the center of the *circumcircle* of the triangle; that is, the circle which passes thru all three vertices of the triangle. The circumcenter of the triangle is usually denoted by O .



Proof. Let $\triangle ABC$ be nondegenerate. Let ℓ and m be perpendicular bisectors to sides $[AB]$ and $[AC]$ correspondingly.

Assume ℓ and m intersect, let $O = \ell \cap m$.

Let us apply Theorem 5.2. Since $O \in \ell$, we have that $OA = OB$ and since $O \in m$, we have that $OA = OC$. It follows that $OB = OC$; that is, O lies on the perpendicular bisector to $[BC]$.

It remains to show that $\ell \nparallel m$; assume the contrary. Since $\ell \perp (AB)$ and $m \perp (AC)$, we get that $(AC) \parallel (AB)$ (see Exercise 7.4). Therefore, by Theorem 5.5, $(AC) = (AB)$; that is, $\triangle ABC$ is degenerate — a contradiction. $\square$

8.2. Exercise. *There is a unique circle which passes thru the vertexes of a given nondegenerate triangle in the Euclidean plane.*

Altitudes and orthocenter

An *altitude* of a triangle is a line thru a vertex and perpendicular to the line containing the opposite side. The term *altitude* maybe also used for the distance from the vertex to its foot point on the line containing opposite side.

8.3. Theorem. *The three altitudes of any nondegenerate triangle intersect in a single point.*

The point of intersection of altitudes is called *orthocenter*; it is usually denoted by H .

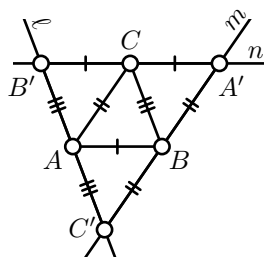
Proof. Let $\triangle ABC$ be nondegenerate.

Consider three lines ℓ , m and n such that

$$\begin{aligned} \ell &\parallel (BC), & m &\parallel (CA), & n &\parallel (AB), \\ \ell &\ni A, & m &\ni B, & n &\ni C. \end{aligned}$$

Since $\triangle ABC$ is nondegenerate, no pair of the lines ℓ , m and n is parallel. Set

$$A' = m \cap n, \quad B' = n \cap \ell, \quad C' = \ell \cap m.$$



Note that $\square ABA'C$, $\square BCB'A$ and $\square CBC'A$ are parallelograms. Applying Lemma 7.16 we get that $\triangle ABC$ is the median triangle of $\triangle A'B'C'$; that is, A , B and C are the midpoints of $[B'C']$, $[C'A']$ and $[A'B']$ correspondingly.

By Exercise 7.4, $(B'C') \parallel (BC)$, the altitude from A is perpendicular to $[B'C']$ and from above it bisects $[B'C']$.

Hence the altitudes of $\triangle ABC$ are also perpendicular bisectors of $\triangle A'B'C'$. Applying Theorem 8.1, we get that altitudes of $\triangle ABC$ intersect at one point. $\square$

8.4. Exercise. *Assume H is the orthocenter of an acute triangle ABC . Show that A is the orthocenter of $\triangle HBC$.*

Medians and centroid

A median of a triangle is the segment joining a vertex to the midpoint of the opposing side.

8.5. Theorem. *The three medians of any nondegenerate triangle intersect in a single point. Moreover, the point of intersection divides each median in the ratio 2:1.*

The point of intersection of medians is called the *centroid* of the triangle; it is usually denoted by M .

Proof. Consider a nondegenerate triangle ABC . Let $[AA']$ and $[BB']$ be its medians.

According to Exercise 3.14, $[AA']$ and $[BB']$ are intersecting. Let us denote the point of intersection by M .

By SAS, $\triangle B'A'C \sim \triangle ABC$ and $A'B' = \frac{1}{2} \cdot AB$. In particular, $\angle ABC = \angle B'A'C$.

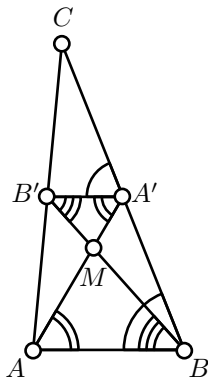
Note that $(AB) \parallel (A'B')$. Indeed, since A' lies between B and C , we have that $\angle BA'B' + \angle B'A'C = \pi$. Therefore,

$$\angle B'A'B + \angle A'BA = \pi;$$

it remains to apply the transversal property 7.6.

Note that A' and A lie on opposite sides from (BB') . Therefore, by the transversal property 7.6, we get that

$$\angle B'A'M = \angle BAM.$$



The same way we get that

$$\angle A'B'M = \angle ABM.$$

By AA condition, $\triangle ABM \sim \triangle A'B'M$. Since $A'B' = \frac{1}{2} \cdot AB$, we have

$$\frac{A'M}{AM} = \frac{B'M}{BM} = \frac{1}{2};$$

that is, M divides medians $[AA']$ and $[BB']$ in ratio 2:1.

Note that M is a unique point on $[BB']$ such that

$$\frac{B'M}{BM} = \frac{1}{2}.$$

Repeating the same argument for vertices B and C we get that all medians $[CC']$ and $[BB']$ intersect at M . $\square$

8.6. Exercise. *Let $\square ABCD$ be a nondegenerate quadrilateral and X, Y, V, W be the midpoints of its sides $[AB], [BC], [CD]$ and $[DA]$. Show that $\square XYVW$ is a parallelogram.*

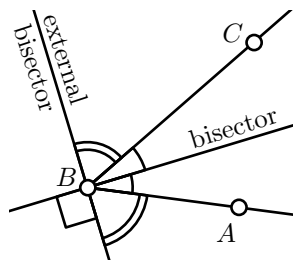
Angle bisectors

If $\angle ABX \equiv -\angle CBX$, then we say that the line (BX) *bisects* $\angle ABC$, or line (BX) is the *bisector* of $\angle ABC$. If $\angle ABX \equiv \pi - \angle CBX$, then the line (BX) is called the *external bisector* of $\angle ABC$.

If $\angle ABA' = \pi$; that is, if B lies between A and A' , then bisector of $\angle ABC$ is the external bisector of $\angle A'BC$ and the other way around.

Note that the bisector and the external bisector are uniquely defined by the angle.

8.7. Exercise. Show that for any angle, its bisector and external bisector are perpendicular.



The bisectors of $\angle ABC$, $\angle BCA$ and $\angle CAB$ of a nondegenerate triangle ABC are called bisectors of $\triangle ABC$ at vertexes A , B and C correspondingly.

8.8. Lemma. Let $\triangle ABC$ be a nondegenerate triangle. Assume that the bisector at the vertex A intersects the side $[BC]$ at the point D . Then

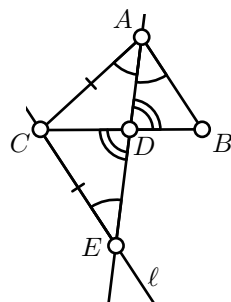
$$\textcircled{1} \quad \frac{AB}{AC} = \frac{DB}{DC}.$$

Proof. Let ℓ be the line passing thru C that is parallel to (AB) . Note that $\ell \nparallel (AD)$; set

$$E = \ell \cap (AD).$$

Also note that B and C lie on opposite sides of (AD) . Therefore, by the transversal property (7.6),

$$\textcircled{2} \quad \angle BAD = \angle CED.$$



Further, note that the angles ADB and EDC are vertical; in particular, by 2.13

$$\angle ADB = \angle EDC.$$

By the AA similarity condition, $\triangle ABD \sim \triangle ECD$. In particular,

$$\textcircled{3} \quad \frac{AB}{EC} = \frac{DB}{DC}.$$

Since (AD) bisects $\angle BAC$, we get that $\angle BAD = \angle DAC$. Together with $\textcircled{2}$, it implies that $\angle CEA = \angle EAC$. By Theorem 4.2, $\triangle ACE$ is isosceles; that is,

$$EC = AC.$$

Together with ❸, it implies ❶. □

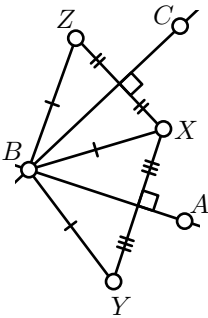
8.9. Exercise. Formulate and prove an analog of Lemma 8.8 for the external bisector.

Equidistant property

Recall that distance from the line ℓ to the point P is defined as the distance from P to its foot point on ℓ ; see page 39.

8.10. Proposition.✓ Assume $\triangle ABC$ is not degenerate. Then a point X lies on the bisector or external bisector of $\angle ABC$ if and only if X is equidistant from the lines (AB) and (BC) .

Proof. We can assume that X does not lie on the union of (AB) and (BC) . Otherwise the distance to one of the lines vanish; in this case $X = B$ is the only point equidistant from the two lines.



Let Y and Z be the reflections of X in (AB) and (BC) correspondingly. Note that

$$Y \neq Z.$$

Otherwise both lines (AB) and (BC) are perpendicular bisectors of $[XY]$, that is, $(AB) = (BC)$ which is impossible since $\triangle ABC$ is not degenerate.

By Proposition 5.6,

$$XB = YB = ZB.$$

Note that X is equidistant from (AB) and (BC) if and only if $XY = XZ$. Applying SSS and then SAS, we get that

$$XY = XZ.$$

$$\Updownarrow$$

$$\triangle BXY \cong \triangle BXZ.$$

$$\Updownarrow$$

$$\angle XBY = \pm \angle BXZ.$$

Since $Y \neq Z$, we get that $\angle XBY \neq \angle BXZ$; therefore

$$\text{❷} \quad \angle XBY = -\angle BXZ.$$

By Proposition 5.6, A lies on the bisector of $\angle XBY$ and B lies on the bisector of $\angle XBZ$; that is,

$$2 \cdot \angle XBA \equiv \angle XBY,$$

$$2 \cdot \angle XBC \equiv \angle XBZ.$$

By ④,

$$2 \cdot \angle XBA \equiv -2 \cdot \angle XBC.$$

The last identity means either

$$\angle XBA + \angle XBC \equiv 0 \quad \text{or} \quad \angle XBA + \angle XBC \equiv \pi,$$

and hence the result. $\square$

Incenter

8.11. Theorem.✓ *The angle bisectors of any nondegenerate triangle intersect at one point.*

The point of intersection of bisectors is called the *incenter* of the triangle; it is usually denoted by I . The point I lies on the same distance from each side. In particular, it is the center of a circle tangent to each side of triangle. This circle is called the *incircle* and its radius is called the *inradius* of the triangle.

Proof. Let $\triangle ABC$ be a nondegenerate triangle.

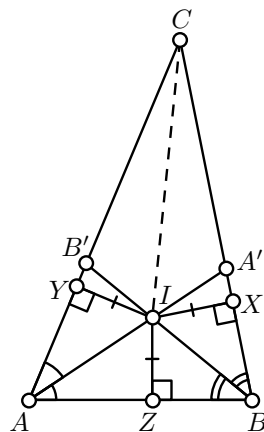
Note that the points B and C lie on opposite sides of the bisector of $\angle BAC$. Hence this bisector intersects $[BC]$ at a point, say A' .

Analogously, there is $B' \in [AC]$ such that (BB') bisects $\angle ABC$.

Applying Pasch's theorem (3.12) twice for the triangles $AA'C$ and $BB'C$, we get that $[AA']$ and $[BB']$ intersect. Let I denotes the point of intersection.

Let X , Y and Z be the foot points of I on (BC) , (CA) and (AB) correspondingly. Applying Proposition 8.10, we get that

$$IY = IZ = IX.$$



From the same lemma, we get that I lies on the bisector or on the exterior bisector of $\angle BCA$.

The line (CI) intersects $[BB']$, the points B and B' lie on opposite sides of (CI) . Therefore, the angles ICB' and ICB have opposite signs. Note that $\angle ICA = \angle ICB'$. Therefore, (CI) cannot be the exterior bisector of $\angle BCA$. Hence the result. $\square$

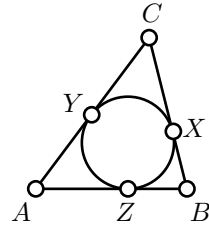
More exercises

8.12. Exercise. Assume that an angle bisector of a nondegenerate triangle bisects the opposite side. Show that the triangle is isosceles.

8.13. Exercise. Assume that at one vertex of a nondegenerate triangle the bisector coincides with the altitude. Show that the triangle is isosceles.

8.14. Exercise. Assume sides $[BC]$, $[CA]$ and $[AB]$ of $\triangle ABC$ are tangent to the incircle at X , Y and Z correspondingly. Show that

$$AY = AZ = \frac{1}{2} \cdot (AB + AC - BC).$$

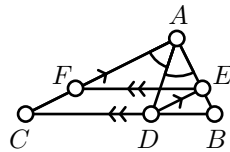


By the definition, the vertexes of *orthic triangle* are the base points of the altitudes of the given triangle.

8.15. Exercise. Prove that the orthocenter of an acute triangle coincides with the incenter of its orthic triangle.

What should be an analog of this statement for an obtuse triangle?

8.16. Exercise. Assume that the bisector at A of the triangle ABC intersects the side $[BC]$ at the point D ; the line thru D and parallel to (CA) intersects (AB) at the point E ; the line thru E and parallel to (BC) intersects (AC) at F . Show that $AE = FC$.



Chapter 9

Inscribed angles

Angle between a tangent line and a chord

9.1. Theorem. Let Γ be a circle with the center O . Assume the line (XQ) is tangent to Γ at X and $[XY]$ is a chord of Γ . Then

$$\textcircled{1} \quad 2 \cdot \angle QXY \equiv \angle XOY.$$

Equivalently,

$$\angle QXY \equiv \frac{1}{2} \cdot \angle XOY \quad \text{or} \quad \angle QXY \equiv \frac{1}{2} \cdot \angle XOY + \pi.$$

Proof. Note that $\triangle XOY$ is isosceles. Therefore, $\angle YXO = \angle OYX$.

Let us apply Theorem 7.10 to $\triangle XOY$. We get

$$\begin{aligned} \pi &\equiv \angle YXO + \angle OYX + \angle XOY \equiv \\ &\equiv 2 \cdot \angle YXO + \angle XOY. \end{aligned}$$

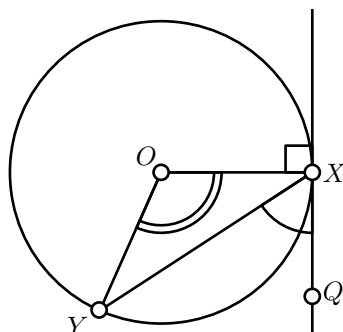
By Lemma 5.16, $(OX) \perp (XQ)$. Therefore,

$$\angle QXY + \angle YXO \equiv \pm \frac{\pi}{2}.$$

Therefore,

$$2 \cdot \angle QXY \equiv \pi - 2 \cdot \angle YXO \equiv \angle XOY.$$

□



Inscribed angle

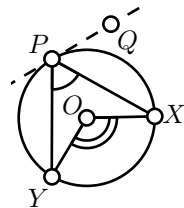
We say that a triangle is *inscribed* in the circle Γ if all its vertices lie on Γ .

9.2. Theorem. Let Γ be a circle with the center O , and X, Y be two distinct points on Γ . Then $\triangle XPY$ is inscribed in Γ if and only if

$$\textcircled{2} \quad 2 \cdot \angle XPY \equiv \angle XOY.$$

Equivalently, if and only if

$$\angle XPY \equiv \frac{1}{2} \cdot \angle XOY \quad \text{or} \quad \angle XPY \equiv \pi + \frac{1}{2} \cdot \angle XOY.$$



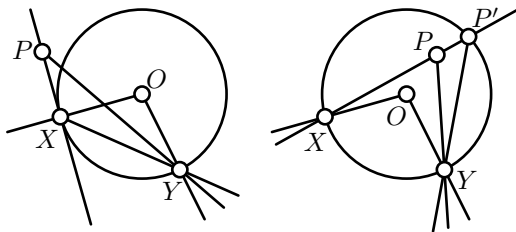
Proof; the “only if” part. Let (PQ) be the tangent line to Γ at P . By Theorem 9.1,

$$2 \cdot \angle QPX \equiv \angle POX, \quad 2 \cdot \angle QPY \equiv \angle POY.$$

Subtracting one identity from the other, we get $\textcircled{2}$.

“If” part. Assume that $\textcircled{2}$ holds for some $P \notin \Gamma$. Note that $\angle XOY \neq 0$. Therefore, $\angle XPY \neq 0$ nor π ; that is, $\triangle PXY$ is nondegenerate.

The line (PX) might be tangent to Γ at the point X or intersect Γ at another point; in the latter case, let P' denotes this point of intersection.



In the first case, by Theorem 9.1, we have

$$2 \cdot \angle PXY \equiv \angle XOY \equiv 2 \cdot \angle XPY.$$

Applying the transversal property (7.6), we get that $(XY) \parallel (PY)$, which is impossible since $\triangle PXY$ is nondegenerate.

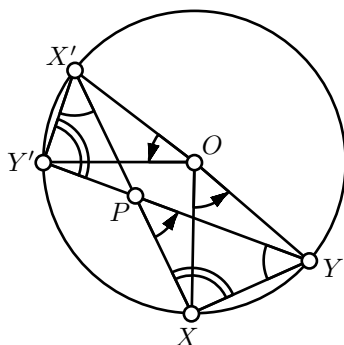
In the second case, applying the “if” part and that P, X and P' lie on one line (see Exercise 2.11) we get that

$$2 \cdot \angle XPP' \equiv 2 \cdot \angle XPY \equiv \angle XOY \equiv 2 \cdot \angle XP'Y \equiv 2 \cdot \angle XP'P.$$

Again, by transversal property, $(PY) \parallel (P'Y)$, which is impossible since $\triangle PXY$ is nondegenerate. $\square$

9.3. Exercise. Let X, X', Y and Y' be distinct points on the circle Γ . Assume (XX') meets (YY') at the point P . Show that

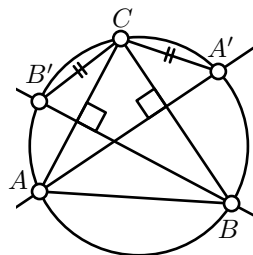
- (a) $2 \cdot \angle XPY = \angle XOY + \angle X'OY'$;
- (b) $\triangle PXY \sim \triangle PY'X'$;
- (c) $PX \cdot PX' = |OP^2 - r^2|$, where O is the center and r is the radius of Γ .



9.4. Exercise. Three chords $[XX']$, $[YY']$ and $[ZZ']$ of the circle Γ intersect at one point. Show that

$$XY' \cdot ZX' \cdot YZ' = X'Y \cdot Z'X \cdot Y'Z.$$

9.5. Exercise. Let Γ be a circumcircle of an acute triangle ABC . Let A' and B' denote the second points of intersection of the altitudes from A and B with Γ . Show that $\triangle A'B'C$ is isosceles.



Recall that the diameter of a circle is its chord which passes thru the center. Note that if $[XY]$ is the diameter of a circle with center O , then $\angle XOY = \pi$. Hence Theorem 9.2 implies the following:

9.6. Corollary. Let Γ be a circle with the diameter $[XY]$. Assume that the point P is distinct from X and Y . Then $P \in \Gamma$ if and only if $\angle XPY$ is right.

9.7. Exercise. Given four points A, B, A' and B' , construct a point Z such that both angles AZB and $A'ZB'$ are right.

9.8. Exercise. Let $\triangle ABC$ be a nondegenerate triangle, A' and B' be foot points of altitudes from A and B respectively. Show that the four points A, B, A' and B' lie on one circle.

What is the center of this circle?

9.9. Exercise. Assume a line ℓ , a circle with its center on ℓ and a point $P \notin \ell$ are given. Make a ruler-only construction of the perpendicular to ℓ from P .

Inscribed quadrilaterals

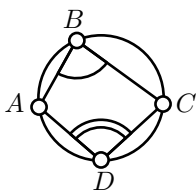
A quadrilateral $ABCD$ is called *inscribed* if all the points A , B , C and D lie on a circle or a line.

9.10. Theorem. *The quadrilateral $ABCD$ is inscribed if and only if*

$$\textcircled{3} \quad 2 \cdot \angle ABC + 2 \cdot \angle CDA \equiv 0.$$

Equivalently, if and only if

$$\angle ABC + \angle CDA \equiv \pi \quad \text{or} \quad \angle ABC \equiv -\angle CDA.$$



Proof of Theorem 9.10. Assume $\triangle ABC$ is degenerate. By Corollary 2.9,

$$2 \cdot \angle ABC \equiv 0;$$

From the same corollary, we get that

$$2 \cdot \angle CDA \equiv 0$$

if and only if $D \in (AB)$; hence the result follows.

It remains to consider the case if $\triangle ABC$ is nondegenerate.

Let O and Γ denote the circulcenter and circumcircle of $\triangle ABC$. According to Theorem 9.2,

$$\textcircled{4} \quad 2 \cdot \angle ABC \equiv \angle AOC.$$

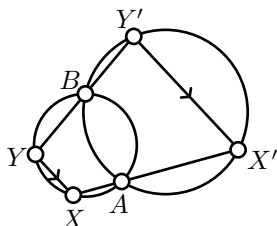
From the same theorem, $D \in \Gamma$ if and only if

$$\textcircled{5} \quad 2 \cdot \angle CDA \equiv \angle COA.$$

Adding $\textcircled{4}$ and $\textcircled{5}$, we get the result. □

9.11. Exercise. *Let $[XY]$ and $[X'Y']$ be two parallel chords of a circle. Show that $XX' = YY'$.*

9.12. Exercise. *Let Γ and Γ' be two circles which intersect at two distinct points: A and B . Assume $[XY]$ and $[X'Y']$ are the chords of Γ and Γ' correspondingly, such that A lies between X and X' and B lies between Y and Y' . Show that $(XY) \parallel (X'Y')$.*



Method of additional circle

9.13. Problem. Assume that two chords $[AA']$ and $[BB']$ intersect at the point P inside their circle. Let X be a point such that both angles XAA' and XBB' are right. Show that $(XP) \perp (A'B')$.

Solution. Set $Y = (A'B') \cap (XP)$.

Both angles XAA' and XBB' are right; therefore

$$2 \cdot \angle XAA' \equiv 2 \cdot \angle XBB'.$$

By Theorem 9.10, $\square XAPB$ is inscribed. Applying this theorem again we get that

$$2 \cdot \angle AXP \equiv 2 \cdot \angle ABP.$$

Since $\square ABA'B'$ is inscribed,

$$2 \cdot \angle ABB' \equiv 2 \cdot \angle AA'B'.$$

It follows that

$$2 \cdot \angle AXY \equiv 2 \cdot \angle AA'Y.$$

By the same theorem $\square XAY A'$ is inscribed, and therefore,

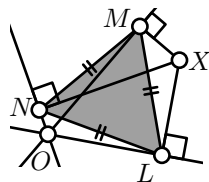
$$2 \cdot \angle XAA' \equiv 2 \cdot \angle XYA'.$$

Since $\angle XAA'$ is right, so is $\angle XYA'$. That is $(XP) \perp (A'B')$. $\square$

9.14. Exercise. Find an inaccuracy in the solution of the problem 9.13 and try to fix it.

The method used in the solution is called *method of additional circle*, since the circumcircles of the $\square XAPB$ and $\square XA'PB'$ above can be considered as *additional constructions*.

9.15. Exercise. Assume three lines ℓ, m and n intersect at point O and form six equal angles at O . Let X be a point distinct from O . Let L, M and N denote the foot points of perpendiculars from X on ℓ, m and n correspondingly. Show that $\triangle LMN$ is equilateral.



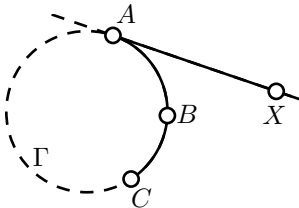
9.16. Advanced exercise. Assume that a point P lies on the circumcircle the triangle ABC . Show that three foot points of P on the lines (AB) , (BC) and (CA) lie on one line (which is called the Simson line of P).

9.17. Exercise. Watch “Why is pi here? And why is it squared? A geometric answer to the Basel problem” by Grant Sanderson. (It is available on YouTube.)

Arcs

A subset of a circle bounded by two points is called a circle arc.

More precisely, let Γ be a circle and A, B, C be distinct points on Γ . The subset which includes the points A, C as well as all the points on Γ which lie with B on the same side of (AC) is called *circle arc ABC* .



For the circle arc ABC , the points A and C are called *endpoints*. There are two circle arcs of Γ with the given endpoints.

A half-line $[AX)$ is called *tangent* to the circle arc ABC at A if the line (AX) is tangent to Γ , and the points X and B lie on the same side of the line (AC) .

If B lies on the line (AC) , then the arc ABC degenerates to one of two following subsets of the line (AC) :

- ◊ If B lies between A and C , then we define the arc ABC as the segment $[AC]$. In this case the half-line $[AC)$ is tangent to the arc ABC at A .
- ◊ If $B \in (AC) \setminus [AC]$, then we define the arc ABC as the line (AC) without all the points between A and C . If we choose points X and $Y \in (AC)$ such that the points X, A, C and Y appear in the same order on the line, then the arc ABC is the union of two half-lines in $[AX)$ and $[CY)$. In this case, the half-line $[AX)$ is tangent to the arc ABC at A .

In addition, any half-line $[AB)$ will be regarded as an arc. This degenerate arc has only one end point A and it assumed to be tangent to itself at A . The circle arcs together with the degenerate arcs will be called *arcs*.

9.18. Proposition. *A point D lies on the arc ABC if and only if*

$$\angle ADC = \angle ABC$$

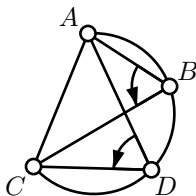
or D coincides with A or C .

Proof. If A, B and C lie on one line, then the statement is evident.

Assume Γ be the circle passing thru A, B and C .

Assume D is distinct from A and C . According to Theorem 9.10, $D \in \Gamma$ if and only if

$$\angle ADC = \angle ABC \quad \text{or} \quad \angle ADC \equiv \angle ABC + \pi.$$



By Exercise 3.13, if the first identity holds, then B and D lie on one side of (AC) ; in this case D belongs to the arc ABC . If the second identity holds, then the points B and

D lie on opposite sides of (AC) , in this case D does not belong to the arc ABC . $\square$

9.19. Proposition. *The half-line $[AX)$ is tangent to the arc ABC if and only if*

$$\angle ABC + \angle CAX \equiv \pi.$$

Proof. For a degenerate arc ABC , the statement is evident. Further we assume the arc ABC is nondegenerate.

Applying theorems 9.1 and 9.2, we get that

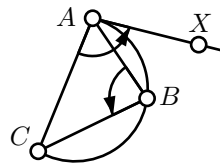
$$2 \cdot \angle ABC + 2 \cdot \angle CAX \equiv 0.$$

Therefore, either

$$\angle ABC + \angle CAX \equiv \pi, \quad \text{or} \quad \angle ABC + \angle CAX \equiv 0.$$

Since $[AX)$ is the tangent half-line to the arc ABC , X and B lie on the same side of (AC) . Therefore, the angles CAX , CAB and ABC have the same sign. In particular, $\angle ABC + \angle CAX \not\equiv 0$; that is, we are left with the case

$$\angle ABC + \angle CAX \equiv \pi. \quad \square$$



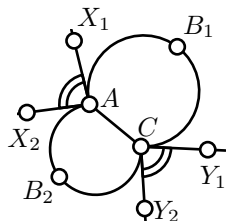
9.20. Exercise. *Assume that the half-lines $[AX)$ and $[AY)$ are tangent to the arcs ABC and ACB correspondingly. Show that $\angle XAY$ is straight.*

9.21. Exercise. *Show that there is a unique arc with endpoints at the given points A and C , which is tangent at A to the given half line $[AX)$.*

9.22. Exercise. *Let $[AX)$ be the tangent half-line to an arc ABC . Assume Y is a point on the arc ABC which is distinct from A . Show that $\angle XAY \rightarrow 0$ as $AY \rightarrow 0$.*

9.23. Exercise. *Given two circle arcs AB_1C and AB_2C , let $[AX_1)$ and $[AX_2)$ be the half-lines tangent to the arcs AB_1C and AB_2C at A , and $[CY_1)$ and $[CY_2)$ be the half-lines tangent to the arcs AB_1C and AB_2C at C . Show that*

$$\angle X_1AX_2 \equiv -\angle Y_1CY_2.$$



9.24. Exercise. *Given an acute triangle ABC make a compass-and-ruler construction of the point Z such that*

$$\angle AZB = \angle BZC = \angle CZA = \pm \frac{2}{3} \cdot \pi$$

Chapter 10

Inversion

Let Ω be the circle with center O and radius r . The *inversion* of a point P in Ω is the point $P' \in [OP)$ such that

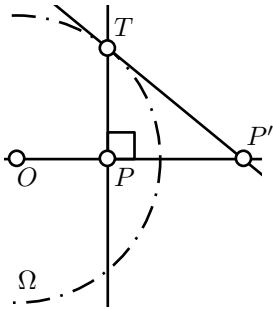
$$OP \cdot OP' = r^2.$$

In this case the circle Ω will be called the *circle of inversion* and its center O is called the *center of inversion*.

The inverse of O is undefined.

Note that if P is inside Ω , then P' is outside and the other way around. Further, $P = P'$ if and only if $P \in \Omega$.

Note that the inversion maps P' back to P .



10.1. Exercise. Let P be a point inside of the circle Ω centered at O . Let T be a point where the perpendicular to (OP) from P intersects Ω . Let P' be the point where the tangent line to Ω at T intersects (OP) .

Show that P' is the inverse of P in the circle Ω .

10.2. Lemma. Let Γ be a circle with the center O . Assume A' and B' are the inverses of A and B in Γ . Then

$$\triangle OAB \sim \triangle OB'A'.$$

Moreover,

$$\angle AOB \equiv -\angle B'OA',$$

$$\angle OBA \equiv -\angle OA'B',$$

$$\angle BAO \equiv -\angle A'B'O.$$

❶

Proof. Let r be the radius of the circle of the inversion.

From the definition of inversion, we get that

$$OA \cdot OA' = OB \cdot OB' = r^2.$$

Therefore,

$$\frac{OA}{OB'} = \frac{OB}{OA'}.$$

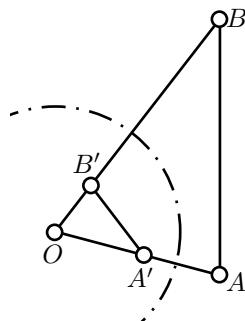
Clearly,

$$\textcircled{2} \quad \angle AOB = \angle A'OB' \equiv -\angle B'OA'.$$

From SAS, we get that

$$\triangle OAB \sim \triangle OB'A'.$$

Applying Theorem 3.7 and $\textcircled{2}$, we get $\textcircled{1}$. □



10.3. Exercise. Let P' be the inverse of P in the circle Γ . Assume that $P \neq P'$. Show that the value $\frac{PX}{P'X}$ is the same for all $X \in \Gamma$.

The converse to the exercise above also holds. Namely, given a positive real number $k \neq 1$ and two distinct points P and P' the locus of points X such that $\frac{PX}{P'X} = k$ forms a circle which is called the *Apollonian circle*. In this case P' is inverse of P in the Apollonian circle.

10.4. Exercise. Let A' , B' and C' be the images of A , B and C under the inversion in the incircle of $\triangle ABC$. Show that the incenter of $\triangle ABC$ is the orthocenter of $\triangle A'B'C'$.

10.5. Exercise. Make a ruler-and-compass construction of the inverse of a given point in a given circle.

Cross-ratio

The following theorem gives some quantities expressed in distances or angles which do not change after inversion.

10.6. Theorem. Let $ABCD$ and $A'B'C'D'$ be two quadrilaterals such that the points A' , B' , C' and D' are the inverses of A , B , C , and D correspondingly.

Then

(a)

$$\frac{AB \cdot CD}{BC \cdot DA} = \frac{A'B' \cdot C'D'}{B'C' \cdot D'A'}.$$

(b)

$$\angle ABC + \angle CDA \equiv -(\angle A'B'C' + \angle C'D'A').$$

(c) If the quadrilateral $ABCD$ is inscribed, then so is $\square A'B'C'D'$.

Proof; (a). Let O be the center of the inversion. According to Lemma 10.2, $\triangle AOB \sim \triangle B'OA'$. Therefore,

$$\frac{AB}{A'B'} = \frac{OA}{OB'}.$$

Analogously,

$$\frac{BC}{B'C'} = \frac{OC}{OB'}, \quad \frac{CD}{C'D'} = \frac{OC}{OD'}, \quad \frac{DA}{D'A'} = \frac{OA}{OD'}.$$

Therefore,

$$\frac{AB}{A'B'} \cdot \frac{B'C'}{BC} \cdot \frac{CD}{C'D'} \cdot \frac{D'A'}{DA} = \frac{OA}{OB'} \cdot \frac{OB'}{OC} \cdot \frac{OC}{OD'} \cdot \frac{OD'}{OA}.$$

Hence (a) follows.

(b). According to Lemma 10.2,

$$\begin{aligned} \textcircled{3} \quad \angle ABO &\equiv -\angle B'A'O, & \angle OBC &\equiv -\angle OC'B', \\ \angle CDO &\equiv -\angle D'C'O, & \angle ODA &\equiv -\angle OA'D'. \end{aligned}$$

By Axiom IIIb,

$$\begin{aligned} \angle ABC &\equiv \angle ABO + \angle OBC, & \angle D'C'B' &\equiv \angle D'C'O + \angle OC'B', \\ \angle CDA &\equiv \angle CDO + \angle ODA, & \angle B'A'D' &\equiv \angle B'A'O + \angle OA'D'. \end{aligned}$$

Therefore, summing the four identities in $\textcircled{3}$, we get that

$$\angle ABC + \angle CDA \equiv -(\angle D'C'B' + \angle B'A'D').$$

Applying Axiom IIIb and Exercise 7.15, we get that

$$\begin{aligned} \angle A'B'C' + \angle C'D'A' &\equiv -(\angle B'C'D' + \angle D'A'B') \equiv \\ &\equiv \angle D'C'B' + \angle B'A'D'. \end{aligned}$$

Hence (b) follows.

(c). Follows from (b) and Theorem 9.10. □

Inversive plane and circlines

Let Ω be a circle with the center O and the radius r . Consider the inversion in Ω .

Recall that inverse of O is undefined. To deal with this problem it is useful to add to the plane an extra point; it will be called the *point at infinity* and we will denote it as ∞ . We can assume that ∞ is inverse of O and the other way around.

The Euclidean plane with an added point at infinity is called the *inversive plane*.

We will always assume that any line and half-line contains ∞ .

It will be convenient to use the notion of *circline*, which means *circle or line*; for instance we may say “if a circline contains ∞ , then it is a line” or “a circline which does not contain ∞ is a circle”.

Note that according to Theorem 8.1, for any $\triangle ABC$ there is a unique circline which passes thru A , B and C (if $\triangle ABC$ is degenerate, then this is a line and if not it is a circle).

10.7. Theorem. *In the inversive plane, inverse of a circline is a circline.*

Proof. Let O denotes the center of the inversion.

Let Γ be a circline. Choose three distinct points A , B and C on Γ . (If $\triangle ABC$ is nondegenerate, then Γ is the circumcircle of $\triangle ABC$; if $\triangle ABC$ is degenerate, then Γ is the line passing thru A , B and C .)

Let A' , B' and C' denote the inverses of A , B and C correspondingly. Let Γ' be the circline which passes thru A' , B' and C' . According to 8.1, Γ' is well defined.

Assume D is a point of the inversive plane which is distinct from A , C , O and ∞ . Let D' denotes the inverse of D .

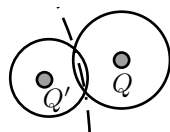
By Theorem 10.6c, $D' \in \Gamma'$ if and only if $D \in \Gamma$. Hence the result.

It remains to prove that $O \in \Gamma \Leftrightarrow \infty \in \Gamma'$ and $\infty \in \Gamma \Leftrightarrow O \in \Gamma'$. Since Γ is the inverse of Γ' , it is sufficient to prove that

$$\infty \in \Gamma \iff O \in \Gamma'.$$

Since $\infty \in \Gamma$, we get that Γ is a line. Therefore, for any $\varepsilon > 0$, the line Γ contains the point P with $OP > r^2/\varepsilon$. For the inversion $P' \in \Gamma'$ of P , we have that $OP' = r^2/OP < \varepsilon$. That is, the circline Γ' contains points arbitrary close to O . It follows that $O \in \Gamma'$. $\square$

10.8. Exercise. *Assume that the circle Γ' is the inverse of the circle Γ . Let Q denotes the center of Γ and Q' denotes the inverse of Q . Show that Q' is not the center of Γ' .*



Assume that a *circumtool* is a geometric construction tool which produces a circline passing thru any three given points.

10.9. Exercise. *Show that with only a circumtool, it is impossible to construct the center of a given circle.*

10.10. Exercise. *Show that for any pair of tangent circles in the inversive plane, there is an inversion which sends them to a pair of parallel lines.*

10.11. Theorem. *Consider the inversion of the inversive plane in the circle Ω with the center O . Then*

- (a) *A line passing thru O is inverted into itself.*
- (b) *A line not passing thru O is inverted into a circle which passes thru O , and the other way around.*
- (c) *A circle not passing thru O is inverted into a circle not passing thru O .*

Proof. In the proof we use Theorem 10.7 without mentioning it.

(a). Note that if a line passes thru O , it contains both ∞ and O . Therefore, its inverse also contains ∞ and O . In particular, the image is a line passing thru O .

(b). Since any line ℓ passes thru ∞ , its image ℓ' has to contain O . If the line does not contain O , then $\ell' \not\ni \infty$; that is, ℓ' is not a line. Therefore, ℓ' is a circle which passes thru O .

(c). If the circle Γ does not contain O , then its image Γ' does not contain ∞ . Therefore, Γ' is a circle. Since $\Gamma \not\ni \infty$ we get that $\Gamma' \not\ni O$. Hence the result. $\square$

Method of inversion

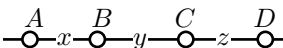
Here is one application of inversion, which we include as an illustration only; we will not use it further in the book.

10.12. Ptolemy's identity. *Let $ABCD$ be an inscribed quadrilateral. Assume that the points A, B, C and D appear on the circline in the same order. Then*

$$AB \cdot CD + BC \cdot DA = AC \cdot BD.$$

Proof. Assume the points A, B, C, D lie on one line in this order.

Set $x = AB$, $y = BC$, $z = CD$. Note that



$$x \cdot z + y \cdot (x + y + z) = (x + y) \cdot (y + z).$$

Since $AC = x + y$, $BD = y + z$ and $DA = x + y + z$, it proves the identity.

It remains to consider the case when the quadrilateral $ABCD$ is inscribed in a circle, say Γ .

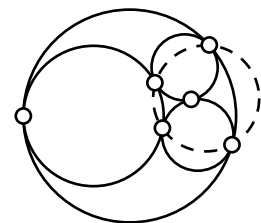
The identity can be rewritten as

$$\frac{AB \cdot DC}{BD \cdot CA} + \frac{BC \cdot AD}{CA \cdot DB} = 1.$$

On the left hand side we have two cross-ratios. According to Theorem 10.6a, the left hand side does not change if we apply an inversion to each point.

Consider an inversion in a circle centered at a point O which lies on Γ between A and D . By Theorem 10.11, this inversion maps Γ to a line. This reduces the problem to the case when A, B, C and D lie on one line, which was already considered. $\square$

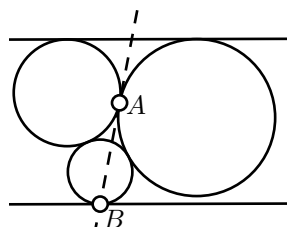
In the proof above, we rewrite Ptolemy's identity in a form which is invariant with respect to inversion and then apply an inversion which makes the statement evident. The solution of the following exercise is based on the same idea; one has to apply an inversion with center at A .



10.13. Exercise. Assume that four circles are mutually tangent to each other. Show that there is four (among six) of their points of tangency lie on one circline.

10.14. Advanced exercise. Assume that three circles tangent to each other and to two parallel lines as shown on the picture.

Show that the line passing thru A and B is also tangent to the two circles at A .



Perpendicular circles

Assume two circles Γ and Ω intersect at two points X and Y . Let ℓ and m be the tangent lines at X to Γ and Ω correspondingly. Analogously, ℓ' and m' be the tangent lines at Y to Γ and Ω .

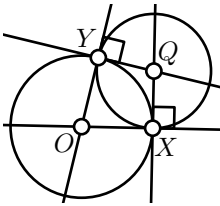
From Exercise 9.23, we get that $\ell \perp m$ if and only if $\ell' \perp m'$.

We say that the circle Γ is *perpendicular* to the circle Ω (briefly $\Gamma \perp \Omega$) if they intersect and the lines tangent to the circle at one point (and therefore, both points) of intersection are perpendicular.

Similarly, we say that the circle Γ is perpendicular to the line ℓ (briefly $\Gamma \perp \ell$) if $\Gamma \cap \ell \neq \emptyset$ and ℓ perpendicular to the tangent lines to Γ at one point (and therefore, both points) of intersection. According to Lemma 5.16, it happens only if the line ℓ passes thru the center of Γ .

Now we can talk about *perpendicular circlines*.

10.15. Theorem. *Assume Γ and Ω are distinct circles. Then $\Omega \perp \Gamma$ if and only if the circle Γ coincides with its inversion in Ω .*



Proof. Let Γ' denotes the inverse of Γ .

“Only if” part. Let O be the center of Ω and Q be the center of Γ . Let X and Y denote the points of intersections of Γ and Ω . According to Lemma 5.16, $\Gamma \perp \Omega$ if and only if (OX) and (OY) are tangent to Γ .

Note that Γ' is also tangent to (OX) and (OY) at X and Y correspondingly. It follows that X and Y are the foot points of the center of Γ' on (OX) and (OY) . Therefore, both Γ' and Γ have the center Q . Finally, $\Gamma' = \Gamma$, since both circles pass thru X .

“If” part. Assume $\Gamma = \Gamma'$.

Since $\Gamma \neq \Omega$, there is a point P which lies on Γ , but not on Ω . Let P' be the inverse of P in Ω . Since $\Gamma = \Gamma'$, we have that $P' \in \Gamma$. In particular, the half-line $[OP)$ intersects Γ at two points. By Exercise 5.12, O lies outside of Γ .

As Γ has points inside and outside of Ω , the circles Γ and Ω intersect. The latter follows from Exercise 3.20.

Let X be a point of their intersection. We need to show that (OX) is tangent to Γ ; that is, X is the only intersection point of (OX) and Γ .

Assume Z is another point of intersection. Since O is outside of Γ , the point Z lies on the half-line $[OX)$.

Let Z' denotes the inverse of Z in Ω . Clearly, the three points Z, Z', X lie on Γ and (OX) . The latter contradicts Lemma 5.14. $\square$

It is convenient to define the *inversion in the line ℓ* as the reflection in ℓ . This way we can talk about *inversion in an arbitrary circline*.

10.16. Corollary. *Let Ω and Γ be distinct circlines in the inversive plane. Then the inversion in Ω sends Γ to itself if and only if $\Omega \perp \Gamma$.*

Proof. By Theorem 10.15, it is sufficient to consider the case when Ω or Γ is a line.

Assume Ω is a line, so the inversion in Ω is a reflection. In this case the statement follows from Corollary 5.7.

If Γ is a line, then the statement follows from Theorem 10.11. $\square$

10.17. Corollary. *Let P and P' be two distinct points such that P' is the inverse of P in the circle Ω . Assume that the circline Γ passes thru P and P' . Then $\Gamma \perp \Omega$.*

Proof. Without loss of generality, we may assume that P is inside and P' is outside Ω . By Theorem 3.17, Γ intersects Ω . Let A denotes a point of intersection.

Let Γ' denotes the inverse of Γ . Since A is a self-inverse, the points A , P and P' lie on Γ' . By Exercise 8.2, $\Gamma' = \Gamma$ and by Theorem 10.15, $\Gamma \perp \Omega$. $\square$

10.18. Corollary. *Let P and Q be two distinct points inside the circle Ω . Then there is a unique circline Γ perpendicular to Ω , which passes thru P and Q .*

Proof. Let P' be the inverse of the point P in the circle Ω . According to Corollary 10.17, the circline is passing thru P and Q is perpendicular to Ω if and only if it passes thru P' .

Note that P' lies outside of Ω . Therefore, the points P , P' and Q are distinct.

According to Exercise 8.2, there is a unique circline passing thru P , Q and P' . Hence the result. $\square$

10.19. Exercise. *Let Ω_1 and Ω_2 be two distinct circles in the Euclidean plane. Assume that the point P does not lie on Ω_1 nor on Ω_2 . Show that there is a unique circline passing thru P which is perpendicular to Ω_1 and Ω_2 .*

10.20. Exercise. *Let P , Q , P' and Q' be points in the Euclidean plane. Assume P' and Q' are inverses of P and Q correspondingly. Show that the quadrilateral $PQP'Q'$ is inscribed.*

10.21. Exercise. *Let Ω_1 and Ω_2 be two perpendicular circles with centers at O_1 and O_2 correspondingly. Show that the inverse of O_1 in Ω_2 coincides with the inverse of O_2 in Ω_1 .*

10.22. Exercise. *Three distinct circles — Ω_1 , Ω_2 and Ω_3 , intersect at two points — A and B . Assume that a circle Γ is perpendicular to Ω_1 and Ω_2 . Show that $\Gamma \perp \Omega_3$.*

10.23. Exercise. Assume you have two construction tools: the circum-tool which constructs a circline thru three given points, and a tool which constructs an inverse of a given point in a given circle.

Assume that a point P does not lie on the two circles Ω_1, Ω_2 . Using only the two given tools, construct a circline Γ that passes thru P , and perpendicular to both Ω_1 and Ω_2 .

Angles after inversion

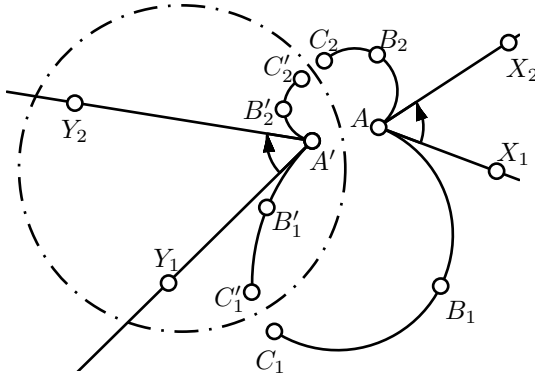
10.24. Proposition. In the inversive plane, the inverse of an arc is an arc.

Proof. Consider four distinct points A, B, C and D ; let A', B', C' and D' be their inverses. We need to show that D lies on the arc ABC if and only if D' lies on the arc $A'B'C'$. According to Proposition 9.18, the latter is equivalent to the following:

$$\angle ADC = \angle ABC \iff \angle A'D'C' = \angle A'B'C'.$$

The latter follows from Theorem 10.6*b*. □

The following theorem states that the angle between arcs changes only its sign after the inversion.

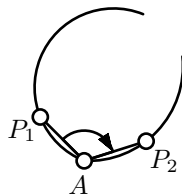


10.25. Theorem. Let AB_1C_1, AB_2C_2 be two arcs in the inversive plane, and the arcs $A'B'_1C'_1, A'B'_2C'_2$ be their inverses. Let $[AX_1)$ and $[AX_2)$ be the half-lines tangent to AB_1C_1 and AB_2C_2 at A , and $[A'Y_1)$ and $[A'Y_2)$ be the half-lines tangent to $A'B'_1C'_1$ and $A'B'_2C'_2$ at A' . Then

$$\angle X_1AX_2 \equiv -\angle Y_1A'Y_2.$$

The *angle between arcs* can be defined as the angle between its tangent half-lines at the common endpoint. Therefore under inversion, the angles between arcs are preserved up to sign.

From Exercise 5.23, it follows that the angle between arcs with common endpoint A is the limit of $\angle P_1AP_2$ where P_1 and P_2 are points approaching A along the corresponding arcs. This observation can be used to define angle between a pair of curves emerging from one point. It turns out that under inversion, angles between curves are also preserved up to sign.



Proof. Applying to Proposition 9.19,

$$\begin{aligned}\angle X_1AX_2 &\equiv \angle X_1AC_1 + \angle C_1AC_2 + \angle C_2AX_2 \equiv \\ &\equiv (\pi - \angle C_1B_1A) + \angle C_1AC_2 + (\pi - \angle AB_2C_2) \equiv \\ &\equiv -(\angle C_1B_1A + \angle AB_2C_2 + \angle C_2AC_1) \equiv \\ &\equiv -(\angle C_1B_1A + \angle AB_2C_1) - (\angle C_1B_2C_2 + \angle C_2AC_1).\end{aligned}$$

The same way we get that

$$\angle Y_1A'Y_2 \equiv -(\angle C'_1B'_1A' + \angle A'B'_2C'_1) - (\angle C'_1B'_2C'_2 + \angle C'_2A'C'_1).$$

By Theorem 10.6b,

$$\begin{aligned}\angle C_1B_1A + \angle AB_2C_1 &\equiv -(\angle C'_1B'_1A' + \angle A'B'_2C'_1), \\ \angle C_1B_2C_2 + \angle C_2AC_1 &\equiv -(\angle C'_1B'_2C'_2 + \angle C'_2A'C'_1)\end{aligned}$$

and hence the result. $\square$

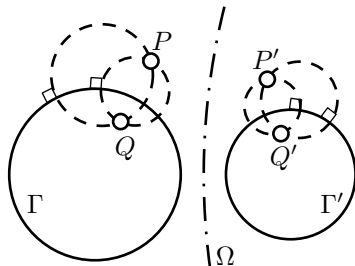
10.26. Corollary. *Let P be the inverse of point Q in a circle Γ . Assume that P' , Q' and Γ' are the inverses of P , Q and Γ in another circle Ω . Then P' is the inverse of Q' in Γ' .*

Proof. If $P = Q$, then $P' = Q' \in \Gamma'$. Therefore, P' is the inverse of Q' in Γ' .

It remains to consider the case $P \neq Q$. Let Δ_1 and Δ_2 be two distinct circles that intersect at P and Q . According to Corollary 10.17, $\Delta_1 \perp \Gamma$ and $\Delta_2 \perp \Gamma$.

Let Δ'_1 and Δ'_2 denote the inverses of Δ_1 and Δ_2 in Ω . Clearly, Δ'_1 meets Δ'_2 at P' and Q' .

From Theorem 10.25, the latter is equivalent to $\Delta'_1 \perp \Gamma'$ and $\Delta'_2 \perp \Gamma'$. By Corollary 10.16, P' is the inverse of Q' in Γ' . $\square$



Chapter 11

Neutral plane

Let us remove Axiom V from our axiomatic system, see page 20. This way we define a new object called *neutral plane* or *absolute plane*. (In a neutral plane, the Axiom V may or may not hold.)

Clearly, any theorem in neutral geometry holds in Euclidean geometry. In other words, the Euclidean plane is an example of a neutral plane. In the next chapter we will construct an example of a neutral plane that is not Euclidean.

In this book, the Axiom V was used for the first time in the proof of uniqueness of parallel lines in Theorem 7.2. Therefore, all the statements before Theorem 7.2 also hold in neutral geometry.

It makes all the discussed results about half-planes, signs of angles, congruence conditions, perpendicular lines and reflections true in neutral geometry. Recall that a statement above marked with “✓”, for example “**Theorem.**✓” if it holds in any neutral plane and the same proof works.

Let us give an example of a theorem in neutral geometry, which admits a simpler proof in Euclidean geometry.

11.1. Hypotenuse-leg congruence condition. *Assume that triangles ABC and $A'B'C'$ have right angles at C and C' correspondingly, $AB = A'B'$ and $AC = A'C'$. Then $\triangle ABC \cong \triangle A'B'C'$.*

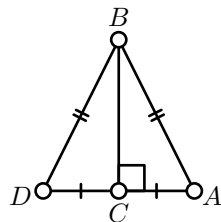
Euclidean proof. By the Pythagorean theorem $BC = B'C'$. Then the statement follows from the SSS congruence condition. $\square$

Note that the proof of the Pythagorean theorem used properties of similar triangles, which in turn used Axiom V. Therefore this proof does not work in a neutral plane.

Neutral proof. Let D denotes the reflection of A in (BC) and D' denotes the reflection of A' in $(B'C')$. Note that

$$AD = 2 \cdot AC = 2 \cdot A'C' = A'D',$$

$$BD = BA = B'A' = B'D'.$$



By SSS congruence condition 4.4, we get that $\triangle ABD \cong \triangle A'B'D'$.

The theorem follows, since C is the midpoint of $[AD]$ and C' is the midpoint of $[A'D']$. $\square$

11.2. Exercise. Give a proof of Exercise 8.12 which works in the neutral plane.

11.3. Exercise. Let $ABCD$ be an inscribed quadrilateral in the neutral plane. Show that

$$\angle ABC + \angle CDA \equiv \angle BCD + \angle DAB.$$

One cannot use the Theorem 9.10 to solve the exercise above, since it uses Theorems 9.1 and 9.2, which in turn uses Theorem 7.10.

Two angles of a triangle

In this section we will prove a weaker form of Theorem 7.10 which holds in any neutral plane.

11.4. Proposition. Let $\triangle ABC$ be a nondegenerate triangle in the neutral plane. Then

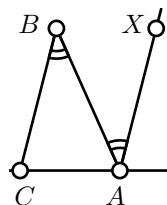
$$|\angle CAB| + |\angle ABC| < \pi.$$

Note that according to 3.7, the angles ABC , BCA and CAB have the same sign. Therefore, in the Euclidean plane the theorem follows immediately from Theorem 7.10.

In neutral geometry, we need to apply Lemma 7.7 directly.

Proof. By Lemma 7.7 there is a point X that lies with B on the same side of (AC) such that $\angle BAX = \angle ABC$. In particular,

❶ $\angle CAX \equiv \angle CAB + \angle ABC.$



By 3.7, the angles CAB , ABC have the same sign and since X and B lie on the same side of (AC) , we also have that the angles CAB and ABX have the same sign. Note that $\angle CAX \neq \pi$, otherwise X would lie on (AC) . Therefore the identity ❶ implies

$$|\angle CAB| + |\angle ABC| = |\angle CAX| < \pi.$$

□

11.5. Exercise. Assume A , B , C and D are points in a neutral plane such that

$$2 \cdot \angle ABC + 2 \cdot \angle BCD \equiv 0.$$

Show that $(AB) \parallel (CD)$.

Note that one cannot apply the transversal property (7.6).

11.6. Exercise. Prove the side-angle-angle congruence condition in the neutral geometry.

In other words, let $\triangle ABC$ and $\triangle A'B'C'$ be two triangles in a neutral plane. Show that $\triangle ABC \cong \triangle A'B'C'$ if

$$AB = A'B', \quad \angle ABC = \pm \angle A'B'C' \quad \text{and} \quad \angle BCA = \pm \angle B'C'A'.$$

Note that in the Euclidean plane, the above exercise follows from ASA and the theorem on the sum of angles of a triangle (7.10). However, Theorem 7.10 cannot be used here, since its proof uses Axiom V. Later (Theorem 13.7) we will show that Theorem 7.10 does not hold in a neutral plane.

11.7. Exercise. Assume that the point D lies between the vertices A and B of $\triangle ABC$ in a neutral plane. Show that

$$CD < CA \quad \text{or} \quad CD < CB.$$

Three angles of triangle

11.8. Proposition. Let $\triangle ABC$ and $\triangle A'B'C'$ be two triangles in the neutral plane such that $AC = A'C'$ and $BC = B'C'$. Then

$$AB < A'B' \quad \text{if and only if} \quad |\angle ACB| < |\angle A'C'B'|.$$

Proof. Without loss of generality, we may assume that $A = A'$ and $C = C'$ and $\angle ACB, \angle ACB' \geq 0$. In this case we need to show that

$$AB < AB' \iff \angle ACB < \angle ACB'.$$

Choose a point X so that

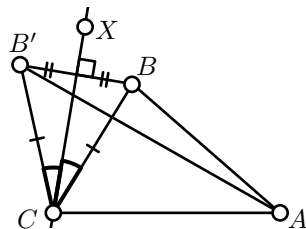
$$\angle ACX = \frac{1}{2} \cdot (\angle ACB + \angle ACB').$$

Note that

- ◇ (CX) bisects $\angle BCB'$.
- ◇ (CX) is the perpendicular bisector of $[BB']$.
- ◇ A and B lie on the same side of (CX) if and only if

$$\angle ACB < \angle ACB'.$$

From Exercise 5.3, A and B lie on the same side of (CX) if and only if $AB < AB'$. Hence the result. $\square$



11.9. Theorem. Let $\triangle ABC$ be a triangle in the neutral plane. Then

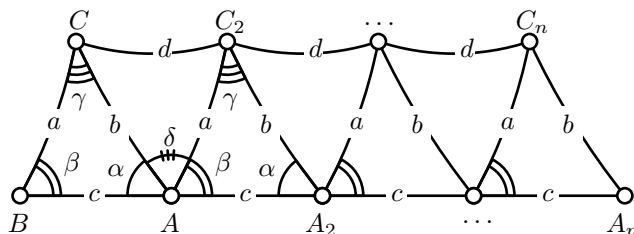
$$|\angle ABC| + |\angle BCA| + |\angle CAB| \leq \pi.$$

The following proof is due to Legendre [12], earlier proofs were due to Saccheri [17] and Lambert [11].

Proof. Let $\triangle ABC$ be the given triangle. Set

$$\begin{aligned} a &= BC, & b &= CA, & c &= AB, \\ \alpha &= \angle CAB, & \beta &= \angle ABC, & \gamma &= \angle BCA. \end{aligned}$$

Without loss of generality, we may assume that $\alpha, \beta, \gamma \geq 0$.



Fix a positive integer n . Consider the points $A_0, A_1, \dots, A_n$ on the half-line $[BA)$, such that $BA_i = i \cdot c$ for each i . (In particular, $A_0 = B$ and $A_1 = A$.) Let us construct the points $C_1, C_2, \dots, C_n$, so that $\angle A_i A_{i-1} C_i = \beta$ and $A_{i-1} C_i = a$ for each i .

By SAS, we have constructed n congruent triangles

$$\begin{aligned}\triangle ABC &= \triangle A_1 A_0 C_1 \cong \\ &\cong \triangle A_2 A_1 C_2 \cong \\ &\quad \dots \\ &\cong \triangle A_n A_{n-1} C_n.\end{aligned}$$

Set $d = C_1 C_2$ and $\delta = \angle C_2 A_1 C_1$. Note that

$$\textcircled{2} \quad \alpha + \beta + \delta = \pi.$$

By Proposition 11.4, we get that $\delta \geq 0$.

By construction

$$\triangle A_1 C_1 C_2 \cong \triangle A_2 C_2 C_3 \cong \dots \cong \triangle A_{n-1} C_{n-1} C_n.$$

In particular, $C_i C_{i+1} = d$ for each i .

By repeated application of the triangle inequality, we get that

$$\begin{aligned}n \cdot c &= A_0 A_n \leq \\ &\leq A_0 C_1 + C_1 C_2 + \dots + C_{n-1} C_n + C_n A_n = \\ &= a + (n-1) \cdot d + b.\end{aligned}$$

In particular,

$$c \leq d + \frac{1}{n} \cdot (a + b - d).$$

Since n is arbitrary positive integer, the latter implies

$$c \leq d.$$

From Proposition 11.8 and SAS, the latter is equivalent to

$$\gamma \leq \delta.$$

From $\textcircled{2}$, the theorem follows. $\square$

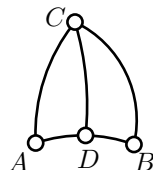
The *defect of triangle* $\triangle ABC$ is defined as

$$\text{defect}(\triangle ABC) := \pi - |\angle ABC| - |\angle BCA| - |\angle CAB|.$$

Note that Theorem 11.9 states that, the defect of any triangle in a neutral plane has to be nonnegative. According to Theorem 7.10, any triangle in the Euclidean plane has zero defect.

11.10. Exercise. Let $\triangle ABC$ be a nondegenerate triangle in the neutral plane. Assume D lies between A and B . Show that

$$\text{defect}(\triangle ABC) = \text{defect}(\triangle ADC) + \text{defect}(\triangle DBC).$$



How to prove that something cannot be proved

Many attempts were made to prove that any theorem in Euclidean geometry holds in neutral geometry. The latter is equivalent to the statement that Axiom V is a *theorem* in neutral geometry.

Some of these attempts were accepted as proofs for long periods of time, until a mistake was found.

There is a number of statements in neutral geometry which are equivalent to the Axiom V. It means that if we exchange the Axiom V to any of these statements, then we will obtain an equivalent axiomatic system.

The following theorem provides a short list of such statements. We are not going to prove it in the book.

11.11. Theorem. *A neutral plane is Euclidean if and only if one of the following equivalent conditions holds:*

- (a) *There is a line ℓ and a point $P \notin \ell$ such that there is only one line passing thru P and parallel to ℓ .*
- (b) *Every nondegenerate triangle can be circumscribed.*
- (c) *There exists a pair of distinct lines that lie on a bounded distance from each other.*
- (d) *There is a triangle with an arbitrary large inradius.*
- (e) *There is a nondegenerate triangle with zero defect.*
- (f) *There exists a quadrilateral in which all the angles are right.*

It is hard to imagine a neutral plane that does not satisfy some of the properties above. That is partly the reason for the large number of false proofs; each used one of such statements by accident.

Let us formulate the negation of (a) above as a new axiom; we label it h-V as a *hyperbolic version* of Axiom V on page 20.

h-V. For any line ℓ and any point $P \notin \ell$ there are at least two lines that pass thru P and parallel to ℓ .

By Theorem 7.2, a neutral plane that satisfies Axiom h-V is not Euclidean. Moreover, according to the Theorem 11.11 (which we do not prove) in any non-Euclidean neutral plane, Axiom h-V holds.

It opens a way to look for a proof by contradiction. Simply exchange Axiom V to Axiom h-V and start to prove theorems in the obtained axiomatic system. In the case if we arrive to a contradiction, we prove the Axiom V in neutral plane. This idea was growing since the 5th century; the most notable results were obtained by Saccheri in [17].

The system of axioms I–IV and h-V defines a new geometry which is now called *hyperbolic* or *Lobachevskian geometry*. The more this geometry was developed, it became more and more believable that there is no contradiction; that is, the system of axioms I–IV and h-V is *consistent*. In fact, the following theorem holds true:

11.12. Theorem. *The hyperbolic geometry is consistent if and only if so is the Euclidean geometry.*

The claims that hyperbolic geometry has no contradiction can be found in private letters of Gauss, Schweikart and Taurinus.¹ They all seem to be afraid to state it in public. For instance, in 1818 Gauss writes to Gerling:

... I am happy that you have the courage to express yourself as if you recognized the possibility that our parallels theory along with our entire geometry could be false. But the wasps whose nest you disturb will fly around your head.

Lobachevsky came to the same conclusion independently. Unlike the others, he had the courage to state it in public and in print (see [13]). That cost him serious troubles. A couple of years later, also independently, Bolyai published his work (see [7]).

It seems that Lobachevsky was the first who had a proof of Theorem 11.12 altho its formulation required rigorous axiomatics, which were not developed at his time. Later, Beltrami gave a cleaner proof of the “if” part of the theorem. It was done by modeling points, lines, distances and angle measures of one geometry using some other objects in another geometry. The same idea was used earlier by Lobachevsky; in [14, 34] he modeled the Euclidean plane in the hyperbolic space.

The proof of Beltrami is the subject of the next chapter.

The discovery of hyperbolic geometry was one of the main scientific discoveries of the 19th century, on the same level are Mendel’s laws and the law of multiple proportions.

¹The oldest surviving letters were the Gauss letter to Gerling in 1816 and yet more convincing letter dated 1818 of Schweikart sent to Gauss via Gerling.

Curvature

In a letter from 1824 Gauss writes:

The assumption that the sum of the three angles is less than π leads to a curious geometry, quite different from ours but completely consistent, which I have developed to my entire satisfaction, so that I can solve every problem in it with the exception of a determination of a constant, which cannot be designated a priori. The greater one takes this constant, the nearer one comes to Euclidean geometry, and when it is chosen indefinitely large the two coincide. The theorems of this geometry appear to be paradoxical and, to the uninitiated, absurd; but calm, steady reflection reveals that they contain nothing at all impossible. For example, the three angles of a triangle become as small as one wishes, if only the sides are taken large enough; yet the area of the triangle can never exceed a definite limit, regardless how great the sides are taken, nor indeed can it ever reach it.

In modern terminology, the constant which Gauss mentions, can be expressed as $1/\sqrt{-k}$, where $k \leq 0$, is the so called *curvature* of the neutral plane, which we are about to introduce.

The identity in Exercise 11.10 suggests that the defect of a triangle should be proportional to its area.²

In fact, for any neutral plane, there is a nonpositive real number k such that

$$k \cdot \text{area}(\triangle ABC) + \text{defect}(\triangle ABC) = 0$$

for any $\triangle ABC$. This number k is called the *curvature* of the plane.

For example, by Theorem 7.10, the Euclidean plane has zero curvature. By Theorem 11.9, the curvature of any neutral plane is nonpositive.

It turns out that up to isometry, the neutral plane is characterized by its curvature; that is, two neutral planes are isometric if and only if they have the same curvature.

In the next chapter, we will construct a *hyperbolic plane*; this is, an example of neutral plane with curvature $k = -1$.

Any neutral planes, distinct from Euclidean, can be obtained by rescaling the metric on the hyperbolic plane. Indeed, if we rescale the metric by a positive factor c , the area changes by factor c^2 , while the defect stays the same. Therefore, taking $c = \sqrt{-k}$, we can get the neutral plane of the given curvature $k < 0$. In other words, all the non-Euclidean neutral planes become identical if we use $r = 1/\sqrt{-k}$ as the unit of length.

²The area in the neutral plane is discussed briefly in the end of Chapter 20, but the reader could also refer to an intuitive understanding of area measurement.

In Chapter 16, we discuss the geometry of the unit sphere. Although spheres are not neutral planes, the spherical geometry is a close relative of Euclidean and hyperbolic geometries.

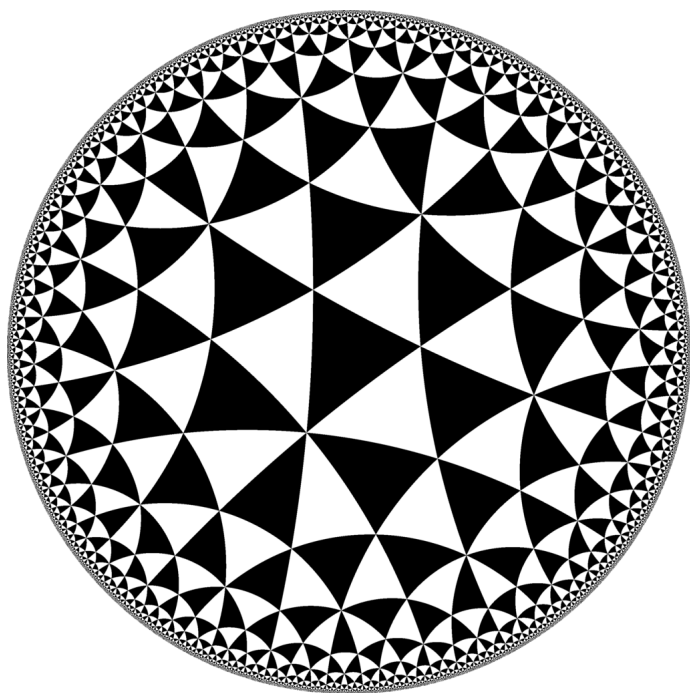
Nondegenerate spherical triangles have negative defects. Moreover, if R is the radius of the sphere, then

$$\frac{1}{R^2} \cdot \text{area}(\triangle ABC) + \text{defect}(\triangle ABC) = 0$$

for any spherical triangle ABC . In other words, the sphere of the radius R has the curvature $k = \frac{1}{R^2}$.

Chapter 12

Hyperbolic plane



In this chapter, we use inversive geometry to construct the model of hyperbolic plane — a neutral plane which is not Euclidean.

Namely, we construct the so called *conformal disc model* of the hyperbolic plane. This model was discovered by Beltrami in [5] and often called the *Poincaré disc model*.

The figure above shows the conformal disc model of the hyperbolic plane which is cut into congruent triangles with angles $\frac{\pi}{3}$, $\frac{\pi}{3}$ and $\frac{\pi}{4}$.

Conformal disc model

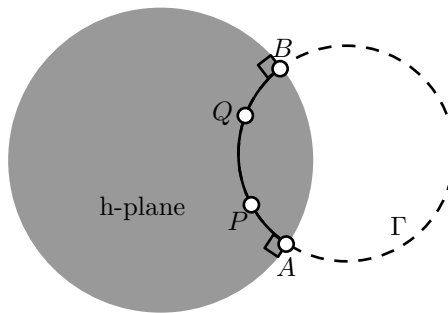
In this section, we give new names for some objects in the Euclidean plane which will represent lines, angle measures and distances in the hyperbolic plane.

Hyperbolic plane. Let us fix a circle on the Euclidean plane and call it *absolute*. The set of points inside the absolute will be called the *hyperbolic plane* (or *h-plane*).

Note that the points on the absolute do *not* belong to the h-plane. The points in the h-plane will be also called *h-points*.

Often we will assume that the absolute is a unit circle.

Hyperbolic lines. The intersections of the h-plane with circlines perpendicular to the absolute are called *hyperbolic lines* or *h-lines*.



By Corollary 10.18, there is a unique h-line which passes thru the given two distinct h-points P and Q . This h-line will be denoted by $(PQ)_h$.

The arcs of hyperbolic lines will be called *hyperbolic segments* or *h-segments*. An h-segment with endpoints P and Q will be denoted by $[PQ]_h$.

The subset of an h-line on one side from a point will be called a *hyperbolic half-line* (or *h-half-line*). More precisely, an h-half-line is an intersection of the h-plane with arc which perpendicular to the absolute with only one endpoint in the h-plane. An h-half-line starting at P and passing thru Q will be denoted by $[PQ]_h$.

If Γ is the circle containing the h-line $(PQ)_h$, then the points of intersection of Γ with the absolute are called *ideal points* of $(PQ)_h$. (Note that the ideal points of an h-line do not belong to the h-line.)

An ordered triple of h-points, say (P, Q, R) will be called *h-triangle* PQR and denoted by $\triangle_h PQR$.

So far, $(PQ)_h$ is just a subset of the h-plane; below we will introduce h-distance and later we will show that $(PQ)_h$ is a line for the h-distance in the sense of the Definition 1.8.

12.1. Exercise. Show that an h-line is uniquely determined by its ideal points.

12.2. Exercise. Show that an h-line is uniquely determined by one of its ideal points and one h-point on it.

12.3. Exercise. Show that the h -segment $[PQ]_h$ coincides with the Euclidean segment $[PQ]$ if and only if the line (PQ) pass thru the center of the absolute.

Hyperbolic distance. Let P and Q be distinct h -points; let A and B denote the ideal points of $(PQ)_h$. Without loss of generality, we may assume that on the Euclidean circle containing the h -line $(PQ)_h$, the points A, P, Q, B appear in the same order.

Consider the function

$$\delta(P, Q) := \frac{AQ \cdot PB}{AP \cdot QB}.$$

Note that the right hand side is a cross-ratio; by Theorem 10.6 it is invariant under inversion. Set $\delta(P, P) = 1$ for any h -point P . Let us define h -distance as the logarithm of δ ; that is,

$$PQ_h := \ln[\delta(P, Q)].$$

The proof that PQ_h is a metric on the h -plane will be given below. For now it is just a function which returns a real value PQ_h for any pair of h -points P and Q .

12.4. Exercise. Let O be the center of the absolute and the h -points O, X and Y lie on one h -line in the same order. Assume $OX = XY$. Prove that $OX_h < XY_h$.

Hyperbolic angles. Consider three h -points P, Q and R such that $P \neq Q$ and $R \neq Q$. The *hyperbolic angle* PQR (briefly $\angle_h PQR$) is an ordered pair of h -half-lines $[QP]_h$ and $[QR]_h$.

Let $[QX)$ and $[QY)$ be (Euclidean) half-lines which are tangent to $[QP]_h$ and $[QR]_h$ at Q . Then the *hyperbolic angle measure* (or *h -angle measure*) of $\angle_h PQR$ denoted by $\angle_h PQR$ and defined as $\angle XQY$.

12.5. Exercise. Let ℓ be an h -line and P be an h -point which does not lie on ℓ . Show that there is a unique h -line passing thru P and perpendicular to ℓ .

Plan of the proof

We defined all the *h -notions* needed in the formulation of the axioms I–IV and h -V. It remains to show that all these axioms hold; this will be done by the end of this chapter.

Once we are done with the proofs, we get that the model provides an example of a neutral plane; in particular, Exercise 12.5 can be proved the same way as Theorem 5.5.

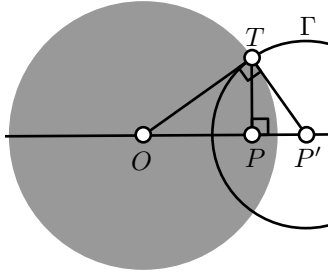
Most importantly we will prove the “if”-part of Theorem 11.12.

Indeed, any statement in hyperbolic geometry can be restated in the Euclidean plane using the introduced h-notions. Therefore, if the system of axioms I–IV and h-V leads to a contradiction, then so does the system axioms I–V.

Auxiliary statements

12.6. Lemma. *Consider an h-plane with a unit circle as the absolute. Let O be the center of the absolute and P be another h-point. Let P' denotes the inverse of P in the absolute.*

Then the circle Γ with the center P' and radius $\frac{\sqrt{1-OP^2}}{OP}$ is perpendicular to the absolute. Moreover, O is the inverse of P in Γ .



Proof. Follows from Exercise 10.21. $\square$

Assume Γ is a circline which is perpendicular to the absolute. Consider the inversion $X \mapsto X'$ in Γ , or if Γ is a line, set $X \mapsto X'$ to be the reflection in Γ .

The following observation says that the map $X \mapsto X'$ respects all the notions introduced in the previous section. Together with the lemma above, it implies that in any problem which is formulated entirely in h-terms we can assume that a given h-point lies in the center of the absolute.

12.7. Main observation. *The map $X \mapsto X'$ described above is a bijection of the h-plane to itself. Moreover, for any h-points P, Q, R such that $P \neq Q$ and $Q \neq R$, the following conditions hold:*

- (a) *The h-line $(PQ)_h$, h-half-line $[PQ]_h$ and h-segment $[PQ]_h$ are transformed into $(P'Q')_h$, $[P'Q']_h$ and $[P'Q']_h$ correspondingly.*
- (b) *$\delta(P', Q') = \delta(P, Q)$ and $P'Q'_h = PQ_h$.*
- (c) *$\angle_h P'Q'R' \equiv -\angle_h PQR$.*

It is instructive to compare this observation with Proposition 5.6.

Proof. According to Theorem 10.15, the map sends the absolute to itself. Note that the points on Γ do not move, it follows that points inside of the absolute remain inside after the mapping and the other way around.

Part (a) follows from 10.7 and 10.25.

Part (b) follows from Theorem 10.6.

Part (c) follows from Theorem 10.25. $\square$

12.8. Exercise. Let Γ be a circle which is perpendicular to the absolute and let Q be an h -point lying on Γ . Assume P is an h -point and P' is its inversion in Γ . Show that $PQ_h = P'Q_h$.

12.9. Lemma. Assume that the absolute is a unit circle centered at O . Given an h -point P , set $x = OP$ and $y = OP_h$. Then

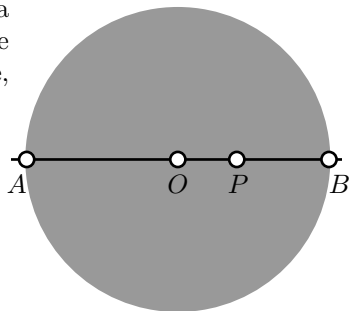
$$y = \ln \frac{1+x}{1-x} \quad \text{and} \quad x = \frac{e^y - 1}{e^y + 1}.$$

Proof. Note that the h -line $(OP)_h$ forms a diameter of the absolute. If A and B are the ideal points as in the definition of h -distance, then

$$OA = OB = 1,$$

$$PA = 1 + x,$$

$$PB = 1 - x.$$



In particular,

$$y = \ln \frac{AP \cdot BO}{PB \cdot OA} = \ln \frac{1+x}{1-x}.$$

Taking the exponential function of the left and the right hand side and applying obvious algebra manipulations, we get that

$$x = \frac{e^y - 1}{e^y + 1}.$$

□

12.10. Lemma. Assume the points P , Q and R appear on one h -line in the same order. Then

$$PQ_h + QR_h = PR_h.$$

Proof. Note that

$$PQ_h + QR_h = PR_h$$

is equivalent to

❶

$$\delta(P, Q) \cdot \delta(Q, R) = \delta(P, R).$$

Let A and B be the ideal points of $(PQ)_h$. Without loss of generality, we can assume that the points A, P, Q, R and B appear in the same order on the circline containing $(PQ)_h$. Then

$$\begin{aligned}\delta(P, Q) \cdot \delta(Q, R) &= \frac{AQ \cdot BP}{QB \cdot PA} \cdot \frac{AR \cdot BQ}{RB \cdot QA} = \\ &= \frac{AR \cdot BP}{RB \cdot PA} = \\ &= \delta(P, R).\end{aligned}$$

Hence ❶ follows. $\square$

Let P be an h-point and $\rho > 0$. The set of all h-points Q such that $PQ_h = \rho$ is called an *h-circle* with the center P and the *h-radius* ρ .

12.11. Lemma. *Any h-circle is a Euclidean circle which lies completely in the h-plane.*

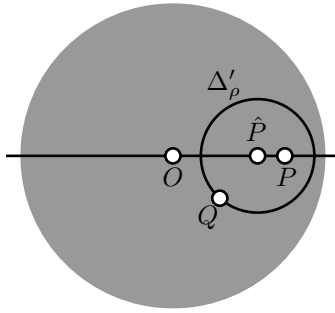
More precisely for any h-point P and $\rho \geq 0$ there is a $\hat{\rho} \geq 0$ and a point $\hat{P}$ such that

$$PQ_h = \rho \iff \hat{P}Q = \hat{\rho}$$

for any h-point Q .

Moreover, if O is the center of the absolute, then

1. $\hat{O} = O$ for any ρ and
2. $\hat{P} \in (OP)$ for any $P \neq O$.



Proof. According to Lemma 12.9, $OQ_h = \rho$ if and only if

$$OQ = \hat{\rho} = \frac{e^\rho - 1}{e^\rho + 1}.$$

Therefore, the locus of h-points Q such that $OQ_h = \rho$ is a Euclidean circle, denote it by Δ_ρ .

If $P \neq O$, applying Lemma 12.6 and the main observation (12.7) we get a circle

Γ perpendicular to the absolute such that P is the inverse of O in Γ .

Let Δ'_ρ be the inverse of Δ_ρ in Γ . Since the inversion in Γ preserves the h-distance, $PQ_h = \rho$ if and only if $Q \in \Delta'_\rho$.

According to Theorem 10.7, Δ'_ρ is a Euclidean circle. Let $\hat{P}$ and $\hat{\rho}$ denote the Euclidean center and radius of Δ'_ρ .

Finally, note that Δ'_ρ reflects to itself in (OP) ; that is, the center $\hat{P}$ lies on (OP) . $\square$

12.12. Exercise. *Assume $P, \hat{P}$ and O are as in the Lemma 12.11 and $P \neq O$. Show that $\hat{P} \in [OP]$.*

Axiom I

Evidently, the h -plane contains at least two points. Therefore, to show that Axiom I holds in the h -plane, we need to show that the h -distance defined on page 91 is a metric on h -plane; that is, the conditions (a)–(d) in Definition 1.1 hold for h -distance.

The following claim says that the h -distance meets the conditions (a) and (b).

12.13. Claim. *Given the h -points P and Q , we have $PQ_h \geq 0$ and $PQ_h = 0$ if and only if $P = Q$.*

Proof. According to Lemma 12.6 and the main observation (12.7), we may assume that Q is the center of the absolute. In this case

$$\delta(Q, P) = \frac{1 + QP}{1 - QP} \geq 1$$

and therefore

$$QP_h = \ln[\delta(Q, P)] \geq 0.$$

Moreover, the equalities holds if and only if $P = Q$. □

The following claim says that the h -distance meets the condition 1.1c.

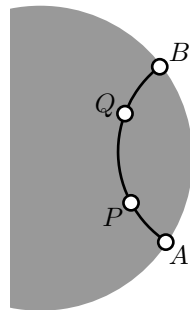
12.14. Claim. *For any h -points P and Q , we have $PQ_h = QP_h$.*

Proof. Let A and B be ideal points of $(PQ)_h$ and A, P, Q, B appear on the circline containing $(PQ)_h$ in the same order.

Then

$$\begin{aligned} PQ_h &= \ln \frac{AQ \cdot BP}{QB \cdot PA} = \\ &= \ln \frac{BP \cdot AQ}{PA \cdot QB} = \\ &= QP_h. \end{aligned}$$

□



The following claim shows, in particular, that the triangle inequality (which is condition 1.1d) holds for h -distance.

12.15. Claim. *Given a triple of h -points P , Q and R , we have*

$$PQ_h + QR_h \geq PR_h.$$

Moreover, the equality holds if and only if P , Q and R lie on one h -line in the same order.

Proof. Without loss of generality, we may assume that P is the center of the absolute and $PQ_h \geq QR_h > 0$.

Let Δ denotes the h-circle with the center Q and h-radius $\rho = QR_h$. Let S and T be the points of intersection of (PQ) and Δ .

By Lemma 12.10, $PQ_h \geq QR_h$. Therefore, we can assume that the points P, S, Q and T appear on the h-line in the same order.

According to Lemma 12.11, Δ is a Euclidean circle; let $\hat{Q}$ denotes its Euclidean center. Note that $\hat{Q}$ is the Euclidean midpoint of $[ST]$.

By the Euclidean triangle inequality

$$\textcircled{2} \quad PT = P\hat{Q} + \hat{Q}R \geq PR$$

and the equality holds if and only if $T = R$.

By Lemma 12.9,

$$PT_h = \ln \frac{1 + PT}{1 - PT},$$

$$PR_h = \ln \frac{1 + PR}{1 - PR}.$$

Since the function $f(x) = \ln \frac{1+x}{1-x}$ is increasing for $x \in [0, 1)$, inequality $\textcircled{2}$ implies

$$PT_h \geq PR_h$$

and the equality holds if and only if $T = R$.

Finally, applying Lemma 12.10 again, we get that

$$PT_h = PQ_h + QR_h.$$

Hence the claim follows. $\square$

Axiom II

Note that once the following claim is proved, Axiom II follows from Corollary 10.18.

12.16. Claim. *A subset of the h-plane is an h-line if and only if it forms a line for the h-distance in the sense of Definition 1.8.*

Proof. Let ℓ be an h-line. Applying the main observation (12.7) we can assume that ℓ contains the center of the absolute. In this case, ℓ is an intersection of a diameter of the absolute and the h-plane. Let A and B be the endpoints of the diameter.

Consider the map $\iota: \ell \rightarrow \mathbb{R}$ defined as

$$\iota(X) = \ln \frac{AX}{XB}.$$

Note that $\iota: \ell \rightarrow \mathbb{R}$ is a bijection.

Further, if $X, Y \in \ell$ and the points A, X, Y and B appear on $[AB]$ in the same order, then

$$\iota(Y) - \iota(X) = \ln \frac{AY}{YB} - \ln \frac{AX}{XB} = \ln \frac{AY \cdot BX}{YB \cdot XB} = XY_h.$$

We proved that any h-line is a line for h-distance. The converse follows from Claim 12.15. $\square$

Axiom III

Note that the first part of Axiom III follows directly from the definition of the h-angle measure defined on page 91. It remains to show that $\angle_h$ satisfies the conditions IIIa, IIIb and IIIc on page 20.

The following two claims say that $\angle_h$ satisfies IIIa and IIIb.

12.17. Claim. *Given an h-half-line $[OP)_h$ and $\alpha \in (-\pi, \pi]$, there is a unique h-half-line $[OQ)_h$ such that $\angle_h POQ = \alpha$.*

12.18. Claim. *For any h-points P, Q and R distinct from an h-point O , we have*

$$\angle_h POQ + \angle_h QOR \equiv \angle_h POR.$$

Proof of 12.17 and 12.18. Applying the main observation, we may assume that O is the center of the absolute. In this case, for any h-point $P \neq O$, $[OP)_h$ is the intersection of $[OP)$ with h-plane. Hence the claims 12.17 and 12.18 follow from the corresponding axioms of the Euclidean plane. $\square$

The following claim says that $\angle_h$ satisfies IIIc.

12.19. Claim. *The function*

$$\angle_h: (P, Q, R) \mapsto \angle_h PQR$$

is continuous at any triple of points (P, Q, R) such that $Q \neq P, Q \neq R$ and $\angle_h PQR \neq \pi$.

Proof. Let O denotes the center of the absolute. We can assume that Q is distinct from O .

Let Z denotes the inverse of Q in the absolute; let Γ denotes the circle perpendicular to the absolute which is centered at Z . According to Lemma 12.6, the point O is the inverse of Q in Γ .

Let P' and R' denote the inversions in Γ of the points P and R correspondingly. Note that the point P' is completely determined by the points Q and P . Moreover, the map $(Q, P) \mapsto P'$ is continuous at any pair of points (Q, P) such that $Q \neq O$. The same is true for the map $(Q, R) \mapsto R'$

According to the main observation

$$\angle_h PQR \equiv -\angle_h P'OR'.$$

Since $\angle_h P'OR' = \angle P'OR'$ and the maps $(Q, P) \mapsto P'$, $(Q, R) \mapsto R'$ are continuous, the claim follows from the corresponding axiom of the Euclidean plane. $\square$

Axiom IV

The following claim says that Axiom IV holds in the h-plane.

12.20. Claim. *In the h-plane, we have $\triangle_h PQR \cong \triangle_h P'Q'R'$ if and only if*

$$Q'P'_h = QP_h, \quad Q'R'_h = QR_h \quad \text{and} \quad \angle_h P'Q'R' = \pm \angle PQR.$$

Proof. Applying the main observation, we can assume that both Q and Q' coincide with the center of the absolute. In this case

$$\angle P'QR' = \angle_h P'QR' = \pm \angle_h PQR = \pm \angle PQR.$$

Since

$$QP_h = QP'_h \quad \text{and} \quad QR_h = QR'_h,$$

Lemma 12.9 implies that the same holds for the Euclidean distances; that is,

$$QP = QP' \quad \text{and} \quad QR = QR'.$$

By SAS, there is a motion of the Euclidean plane which sends Q to itself, P to P' and R to R'

Note that the center of the absolute is fixed by the corresponding motion. It follows that this motion gives also a motion of the h-plane; in particular, the h-triangles $\triangle_h PQR$ and $\triangle_h P'QR'$ are h-congruent. $\square$

Axiom h-V

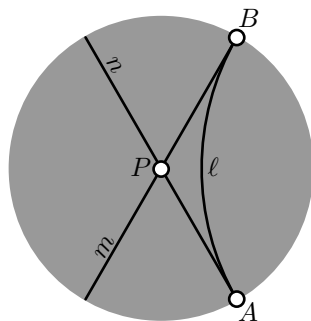
Finally, we need to check that the Axiom h-V on page 85 holds; that is, we need to prove the following claim.

12.21. Claim. *For any h-line ℓ and any h-point $P \notin \ell$ there are at least two h-lines which pass thru P and have no points of intersection with ℓ .*

Instead of proof. Applying the main observation we can assume that P is the center of the absolute.

The remaining part of the proof can be guessed from the picture $\square$

12.22. Exercise. Show that in the h -plane there are 3 mutually parallel h -lines such that any pair of these three lines lies on one side of the remaining h -line.



Hyperbolic trigonometry

In this section we give formulas for h -distance using *hyperbolic functions*. One of these formulas will be used in the proof of the hyperbolic Pythagorean theorem (13.13).

Recall that ch , sh and th denote *hyperbolic cosine*, *hyperbolic sine* and *hyperbolic tangent*; that is, the functions defined by

$$\text{ch } x := \frac{e^x + e^{-x}}{2}, \quad \text{sh } x := \frac{e^x - e^{-x}}{2},$$

$$\text{th } x := \frac{\text{sh } x}{\text{ch } x}.$$

These hyperbolic functions are analogous to sine and cosine and tangent.

12.23. Exercise. Prove the following identities:

$$\text{ch}' x = \text{sh } x; \quad \text{sh}' x = \text{ch } x; \quad (\text{ch } x)^2 - (\text{sh } x)^2 = 1.$$

12.24. Double-argument identities. The identities

$$\text{ch}(2 \cdot x) = (\text{ch } x)^2 + (\text{sh } x)^2 \quad \text{and} \quad \text{sh}(2 \cdot x) = 2 \cdot \text{sh } x \cdot \text{ch } x$$

hold for any real value x .

Proof.

$$\begin{aligned} (\text{sh } x)^2 + (\text{ch } x)^2 &= \left(\frac{e^x - e^{-x}}{2}\right)^2 + \left(\frac{e^x + e^{-x}}{2}\right)^2 = \\ &= \frac{e^{2 \cdot x} + e^{-2 \cdot x}}{2} = \\ &= \text{ch}(2 \cdot x); \\ 2 \cdot \text{sh } x \cdot \text{ch } x &= 2 \cdot \left(\frac{e^x - e^{-x}}{2}\right) \cdot \left(\frac{e^x + e^{-x}}{2}\right) = \\ &= \frac{e^{2 \cdot x} - e^{-2 \cdot x}}{2} = \\ &= \text{sh}(2 \cdot x). \end{aligned}$$

$\square$

12.25. Advanced exercise. Let P and Q be two h -points distinct from the center of absolute. Denote by P' and Q' the inverses of P and Q in the absolute.

Show that

(a)

$$\operatorname{ch}[\tfrac{1}{2} \cdot PQ_h] = \sqrt{\frac{PQ' \cdot P'Q}{PP' \cdot QQ'}};$$

(b)

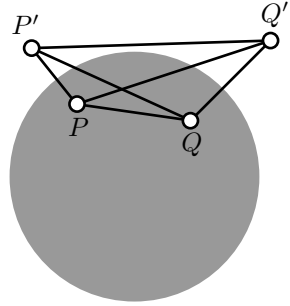
$$\operatorname{sh}[\tfrac{1}{2} \cdot PQ_h] = \sqrt{\frac{PQ \cdot P'Q'}{PP' \cdot QQ'}};$$

(c)

$$\operatorname{th}[\tfrac{1}{2} \cdot PQ_h] = \sqrt{\frac{PQ \cdot P'Q'}{PQ' \cdot P'Q}};$$

(d)

$$\operatorname{ch} PQ_h = \frac{PQ \cdot P'Q' + PQ' \cdot P'Q}{PP' \cdot QQ'}.$$



Chapter 13

Geometry of the h-plane

In this chapter, we study the geometry of the plane described by the conformal disc model. For briefness, this plane will be called the *h-plane*.

We can work with this model directly from inside of the Euclidean plane. We may also use the axioms of neutral geometry since they all hold in the h-plane; the latter proved in the previous chapter.

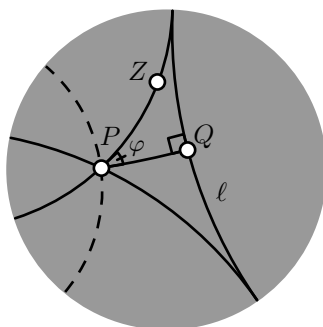
Angle of parallelism

Let P be a point off an h-line ℓ . Drop a perpendicular $(PQ)_h$ from P to ℓ ; let Q be its foot point. Let φ be the smallest value such that the h-line $(PZ)_h$ with $|\angle_h QPZ| = \varphi$ does not intersect ℓ .

The value φ is called the *angle of parallelism* of P to ℓ . Clearly, φ depends only on the h-distance $s = PQ_h$. Further, $\varphi(s) \rightarrow \pi/2$ as $s \rightarrow 0$, and $\varphi(s) \rightarrow 0$ as $s \rightarrow \infty$. (In the Euclidean geometry, the angle of parallelism is identically equal to $\pi/2$.)

If ℓ , P and Z are as above, then the h-line $m = (PZ)_h$ is called *asymptotically parallel* to ℓ . In other words, two h-lines are asymptotically parallel if they share one ideal point. (In hyperbolic geometry, the term *parallel lines* is often used for *asymptotically parallel lines*; we do not follow this convention.)

Given $P \notin \ell$, there are exactly two asymptotically parallel lines thru P to ℓ ; the remaining parallel lines are called *ultra parallel*.



Proof. Let φ denotes the angle of parallelism of B to $(AC)_h$. Note that $\varphi > \frac{\beta}{2}$; therefore

$$\frac{1}{2} \cdot \ln \frac{1 + \cos \varphi}{1 - \cos \varphi} < \frac{1}{2} \cdot \ln \frac{1 + \cos \frac{\beta}{2}}{1 - \cos \frac{\beta}{2}}.$$

It remains to apply Proposition 13.1. □

13.3. Exercise. Let ABC be an equilateral h -triangle with side 100. Show that $|\angle_h ABC| < 10^{-10}$.

Inradius of h -triangle

13.4. Theorem. The inradius of any h -triangle is less than $\frac{1}{2} \cdot \ln 3$.

Proof. Let I and r be the h -incenter and h -inradius of $\triangle_h XYZ$.

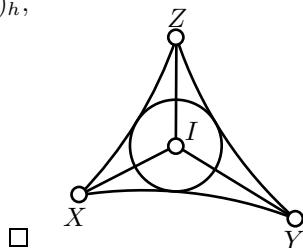
Note that the h -angles XIY , YIZ and ZIX have the same sign. Without loss of generality, we can assume that all of them are positive and therefore

$$\angle_h XIY + \angle_h YIZ + \angle_h ZIX = 2 \cdot \pi$$

We can assume that $\angle_h XIY \geq \frac{2}{3} \cdot \pi$; if not relabel X , Y and Z .

Since r is the h -distance from I to $(XY)_h$, Corollary 13.2 implies that

$$\begin{aligned} r &< \frac{1}{2} \cdot \ln \frac{1 + \cos \frac{\pi}{3}}{1 - \cos \frac{\pi}{3}} = \\ &= \frac{1}{2} \cdot \ln \frac{1 + \frac{1}{2}}{1 - \frac{1}{2}} = \\ &= \frac{1}{2} \cdot \ln 3. \end{aligned}$$



13.5. Exercise. Let $\square_h ABCD$ be a quadrilateral in the h -plane such that the h -angles at A , B and C are right and $AB_h = BC_h$. Find the optimal upper bound for AB_h .

Circles, horocycles and equidistants

Note that according to Lemma 12.11, any h -circle is a Euclidean circle which lies completely in the h -plane. Further, any h -line is an intersection of the h -plane with the circle perpendicular to the absolute.

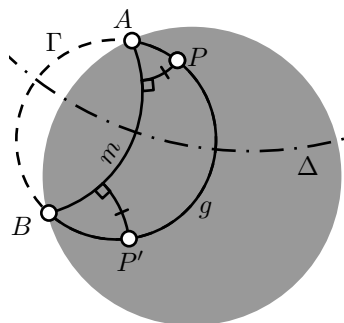
In this section we will describe the h -geometric meaning of the intersections of the other circles with the h -plane.

You will see that all these intersections have a *perfectly round shape* in the h-plane.

One may think of these curves as trajectories of a car which drives in the plane with a fixed position of the steering wheel.

In the Euclidean plane, this way you either run along a circle or along a line.

In the hyperbolic plane, the picture is different. If you turn the steering wheel to the far right, you will run along a circle. If you turn it less, at a certain position of the wheel, you will never come back to the same point, but the path will be different from the line. If you turn the wheel further a bit, you start to run along a path which stays at some fixed distance from an h-line.



Equidistants of h-lines. Consider the h-plane with the absolute Ω . Assume a circle Γ intersects Ω in two distinct points, A and B . Let g denotes the intersection of Γ with the h-plane.

Let us draw an h-line m with the ideal points A and B . According to Exercise 12.1, m is uniquely defined.

Consider any h-line ℓ perpendicular to m ; let Δ be the circle containing ℓ .

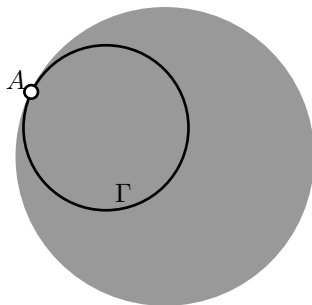
Note that $\Delta \perp \Gamma$. Indeed, according to Corollary 10.16, m and Ω invert to themselves in Δ . It follows that A is the inverse of B in Δ . Finally, by Corollary 10.17, we get that $\Delta \perp \Gamma$.

Therefore, inversion in Δ sends both m and g to themselves. For any two points $P', P \in g$ there is a choice of ℓ and Δ as above such that P' is the invese of P in Δ . By the main observation (12.7) the inversion in Δ is a motion of the h-plane. Therefore, all points of g lie on the same distance from m .

In other words, g is the set of points which lie on a fixed h-distance and on the same side of m .

Such a curve g is called *equidistant* to h-line m . In Euclidean geometry, the equidistant from a line is a line; apparently in hyperbolic geometry the picture is different.

Horocycles. If the circle Γ touches the absolute from inside at one point A , then the complement $h = \Gamma \setminus \{A\}$ lies in the h-plane. This set is called a *horocycle*. It also has a perfectly round shape in the sense described above.



Horocycles are the border case between circles and equidistants to h-lines. A horocycle might be considered as a limit of circles thru a fixed point with the centers running to infinity along a line. The same horocycle is a limit of equidistants which pass thru fixed point to the h-lines running to infinity.

13.6. Exercise. Find the leg of an isosceles right h-triangle inscribed in a horocycle.

Hyperbolic triangles

13.7. Theorem. Any nondegenerate hyperbolic triangle has a positive defect.

Proof. Fix an h-triangle ABC . According to Theorem 11.9,

$$\textcircled{1} \quad \text{defect}(\triangle_h ABC) \geq 0.$$

It remains to show that in the case of equality, $\triangle_h ABC$ degenerates.

Without loss of generality, we may assume that A is the center of the absolute; in this case $\angle_h CAB = \angle CAB$. Yet we may assume that

$$\angle_h CAB, \quad \angle_h ABC, \quad \angle_h BCA, \quad \angle ABC, \quad \angle BCA \geq 0.$$

Let D be an arbitrary point in $[CB]_h$ distinct from B and C . From Proposition 9.19, we have

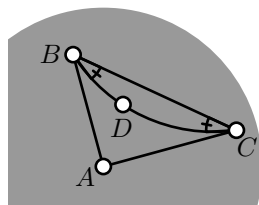
$$\angle ABC - \angle_h ABC \equiv \pi - \angle CDB \equiv \angle BCA - \angle_h BCA.$$

From Exercise 7.12, we get that

$$\text{defect}(\triangle_h ABC) = 2 \cdot (\pi - \angle CDB).$$

Therefore, if equality holds in $\textcircled{1}$, then $\angle CDB = \pi$. In particular, the h-segment $[BC]_h$ coincides with the Euclidean segment $[BC]$. By Exercise 12.3, the latter can happen only if the h-line $(BC)_h$ passes thru the center of the absolute (A); that is, if $\triangle_h ABC$ degenerates. $\square$

The following theorem states, in particular, that nondegenerate hyperbolic triangles are congruent if their corresponding angles are equal. In particular, in hyperbolic geometry, similar triangles have to be congruent.



13.8. AAA congruence condition. *Two nondegenerate h-triangles ABC and $A'B'C'$ are congruent if $\angle_h ABC = \pm \angle_h A'B'C'$, $\angle_h BCA = \pm \angle_h B'C'A'$ and $\angle_h CAB = \pm \angle_h C'A'B'$.*

Proof. Note that if $AB_h = A'B'_h$, then the theorem follows from ASA.

Assume the contrary. Without loss of generality, we may assume that $AB_h < A'B'_h$. Therefore, we can choose the point $B'' \in [A'B']_h$ such that $A'B''_h = AB_h$.

Choose an h-half-line $[B''X)$ so that

$$\angle_h A'B''X = \angle_h A'B'C'.$$

According to Exercise 11.5, $(B''X)_h \parallel (B'C')_h$.

By Pasch's theorem (3.12), $(B''X)_h$ intersects $[A'C']_h$. Let C'' denotes the point of intersection.

According to ASA, $\triangle_h ABC \cong \triangle_h A'B''C''$; in particular,

$$\textcircled{2} \quad \text{defect}(\triangle_h ABC) = \text{defect}(\triangle_h A'B''C'').$$

Applying Exercise 11.10 twice, we get that

$$\begin{aligned} \textcircled{3} \quad \text{defect}(\triangle_h A'B'C') &= \text{defect}(\triangle_h A'B''C'') + \\ &+ \text{defect}(\triangle_h B''C''C') + \text{defect}(\triangle_h B''C'B'). \end{aligned}$$

By Theorem 13.7, all the defects have to be positive. Therefore

$$\text{defect}(\triangle_h A'B'C') > \text{defect}(\triangle_h ABC).$$

On the other hand,

$$\begin{aligned} \text{defect}(\triangle_h A'B'C') &= |\angle_h A'B'C'| + |\angle_h B'C'A'| + |\angle_h C'A'B'| = \\ &= |\angle_h ABC| + |\angle_h BCA| + |\angle_h CAB| = \\ &= \text{defect}(\triangle_h ABC) \end{aligned}$$

— a contradiction. □

Recall that a bijection from a h-plane to itself is called *angle preserving* if

$$\angle_h ABC = \angle_h A'B'C'$$

for any $\triangle_h ABC$ and its image $\triangle_h A'B'C'$.

13.9. Exercise. *Show that any angle-preserving transformation of the h-plane is a motion.*

Conformal interpretation

Let us give another interpretation of the h-distance.

13.10. Lemma. *Consider the h-plane with the unit circle centered at O as the absolute. Fix a point P and let Q be another point in the h-plane. Set $x = PQ$ and $y = PQ_h$. Then*

$$\lim_{x \rightarrow 0} \frac{y}{x} = \frac{2}{1 - OP^2}.$$

The above formula tells us that the h-distance from P to a nearby point Q is almost proportional to the Euclidean distance with the coefficient $\frac{2}{1 - OP^2}$. The value $\lambda(P) = \frac{2}{1 - OP^2}$ is called the *conformal factor* of the h-metric.

The value $\frac{1}{\lambda(P)} = \frac{1}{2} \cdot (1 - OP^2)$ can be interpreted as the *speed limit* at the given point P . In this case the h-distance is the minimal time needed to travel from one point of the h-plane to another point.

Proof. If $P = O$, then according to Lemma 12.9

$$\textcircled{4} \quad \frac{y}{x} = \frac{\ln \frac{1+x}{1-x}}{x} \rightarrow 2$$

as $x \rightarrow 0$.

If $P \neq O$, let P' denotes the inverse of P in the absolute. Let Γ denotes the circle with the center P' perpendicular to the absolute.

According to the main observation (12.7) and Lemma 12.6, the inversion in Γ is a motion of the h-plane which sends P to O . In particular, if Q' denotes the inverse of Q in Γ , then $OQ'_h = PQ_h$.

Set $x' = OQ'$. According to Lemma 10.2,

$$\frac{x'}{x} = \frac{OP'}{P'Q}.$$

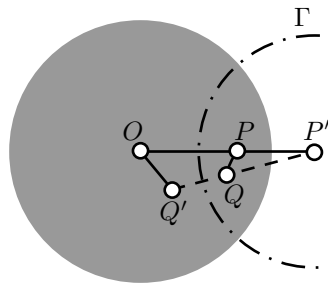
Since P' is the inverse of P in the absolute, we have that $PO \cdot OP' = 1$. Therefore,

$$\frac{x'}{x} \rightarrow \frac{OP'}{P'P} = \frac{1}{1 - OP^2}$$

as $x \rightarrow 0$.

Together with $\textcircled{4}$, it implies that

$$\frac{y}{x} = \frac{y}{x'} \cdot \frac{x'}{x} \rightarrow \frac{2}{1 - OP^2}$$



as $x \rightarrow 0$. □

Here is an application of the lemma above.

13.11. Proposition. *The circumference of an h-circle of the h-radius r is*

$$2 \cdot \pi \cdot \operatorname{sh} r,$$

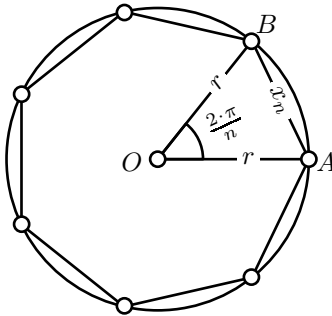
where $\operatorname{sh} r$ denotes the hyperbolic sine of r ; that is,

$$\operatorname{sh} r := \frac{e^r - e^{-r}}{2}.$$

Before we proceed with the proof, let us discuss the same problem in the Euclidean plane.

The circumference of a circle in the Euclidean plane can be defined as the limit of perimeters of regular n -gons inscribed in the circle as $n \rightarrow \infty$.

Namely, let us fix $r > 0$. Given a positive integer n , consider $\triangle AOB$ such that $\angle AOB = \frac{2 \cdot \pi}{n}$ and $OA = OB = r$. Set $x_n = AB$. Note that x_n is the side of a regular n -gon inscribed in the circle of radius r . Therefore, the perimeter of the n -gon is $n \cdot x_n$.



The circumference of the circle with the radius r might be defined as the limit

$$\textcircled{5} \quad \lim_{n \rightarrow \infty} n \cdot x_n = 2 \cdot \pi \cdot r.$$

(This limit can be taken as the definition of π .)

In the following proof, we repeat the same construction in the h-plane.

Proof. Without loss of generality, we can assume that the center O of the circle is

the center of the absolute.

By Lemma 12.9, the h-circle with the h-radius r is the Euclidean circle with the center O and the radius

$$a = \frac{e^r - 1}{e^r + 1}.$$

Let x_n and y_n denote the side lengths of the regular n -gons inscribed in the circle in the Euclidean and hyperbolic plane correspondingly.

Note that $x_n \rightarrow 0$ as $n \rightarrow \infty$. By Lemma 13.10,

$$\lim_{n \rightarrow \infty} \frac{y_n}{x_n} = \frac{2}{1 - a^2}.$$

Applying ⑤, we get that the circumference of the h -circle can be found the following way:

$$\begin{aligned}
 \lim_{n \rightarrow \infty} n \cdot y_n &= \frac{2}{1 - a^2} \cdot \lim_{n \rightarrow \infty} n \cdot x_n = \\
 &= \frac{4 \cdot \pi \cdot a}{1 - a^2} = \\
 &= \frac{4 \cdot \pi \cdot \left(\frac{e^r - 1}{e^r + 1} \right)}{1 - \left(\frac{e^r - 1}{e^r + 1} \right)^2} = \\
 &= 2 \cdot \pi \cdot \frac{e^r - e^{-r}}{2} = \\
 &= 2 \cdot \pi \cdot \operatorname{sh} r.
 \end{aligned}$$

□

13.12. Exercise. Let $\operatorname{circum}_h(r)$ denotes the circumference of the h -circle of the h -radius r . Show that

$$\operatorname{circum}_h(r + 1) > 2 \cdot \operatorname{circum}_h(r)$$

for all $r > 0$.

Pythagorean theorem

Recall that ch denotes *hyperbolic cosine*; that is, the function defined by

$$\operatorname{ch} x := \frac{e^x + e^{-x}}{2}.$$

13.13. Hyperbolic Pythagorean theorem. Assume that ACB is an h -triangle with right angle at C . Set

$$a = BC_h, \quad b = CA_h \quad \text{and} \quad c = AB_h.$$

Then

$$\textcircled{6} \quad \operatorname{ch} c = \operatorname{ch} a \cdot \operatorname{ch} b.$$

The formula ⑥ will be proved by means of direct calculations. Let us discuss the limit cases of this formula.

Note that $\operatorname{ch} x$ can be written using the Taylor expansion

$$\operatorname{ch} x = 1 + \frac{1}{2} \cdot x^2 + \frac{1}{24} \cdot x^4 + \dots$$

It follows that if a , b and c are small, then

$$\begin{aligned} 1 + \frac{1}{2} \cdot c^2 &\approx \operatorname{ch} c = \operatorname{ch} a \cdot \operatorname{ch} b \approx \\ &\approx (1 + \frac{1}{2} \cdot a^2) \cdot (1 + \frac{1}{2} \cdot b^2) \approx \\ &\approx 1 + \frac{1}{2} \cdot (a^2 + b^2). \end{aligned}$$

In other words, the original Pythagorean theorem (6.4) is a limit case of the hyperbolic Pythagorean theorem for small triangles.

For large a and b the terms e^{-a} and e^{-b} are neglectable. In this case we have the following approximations:

$$\begin{aligned} \operatorname{ch} a \cdot \operatorname{ch} b &\approx \frac{e^a}{2} \cdot \frac{e^b}{2} = \\ &= \frac{e^{a+b-\ln 2}}{2} \approx \\ &\approx \operatorname{ch}(a + b - \ln 2). \end{aligned}$$

Therefore $c \approx a + b - \ln 2$.

13.14. Exercise. Assume that ACB is an h -triangle with right angle at C . Set $a = BC_h$, $b = CA_h$ and $c = AB_h$. Show that

$$c + \ln 2 > a + b.$$

In the proof of the hyperbolic Pythagorean theorem we use the following formula from Exercise 12.25:

$$\operatorname{ch} PQ_h = \frac{PQ \cdot P'Q' + PQ' \cdot P'Q}{PP' \cdot QQ'},$$

here P , Q are h -points distinct from the absolute and P' , Q' are their inversions in the absolute. A complete proof of this formula is given in the hints.

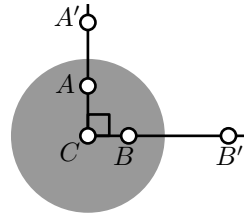
Proof of the hyperbolic Pythagorean theorem. We assume that absolute is a unit circle. By the main observation (12.7) we can assume that C is the center of absolute. Let A' and B' denote the inverses of A and B in the absolute.

Set $x = BC$, $y = AC$. By Lemma 12.9

$$a = \ln \frac{1+x}{1-x}, \quad b = \ln \frac{1+y}{1-y}.$$

Therefore

$$\begin{aligned} \textcircled{7} \quad \operatorname{ch} a &= \frac{1}{2} \cdot \left(\frac{1+x}{1-x} + \frac{1-x}{1+x} \right) = & \operatorname{ch} b &= \frac{1}{2} \cdot \left(\frac{1+y}{1-y} + \frac{1-y}{1+y} \right) = \\ &= \frac{1+x^2}{1-x^2}, & &= \frac{1+y^2}{1-y^2}. \end{aligned}$$



Note that

$$B'C = \frac{1}{x}, \quad A'C = \frac{1}{y}.$$

Therefore

$$BB' = \frac{1}{x} - x, \quad AA' = \frac{1}{y} - y.$$

Since the triangles ABC , $A'BC$, $AB'C$, $A'B'C$ are right, the original Pythagorean theorem (6.4) implies

$$\begin{aligned} AB &= \sqrt{x^2 + y^2}, & AB' &= \sqrt{\frac{1}{x^2} + y^2}, \\ A'B &= \sqrt{x^2 + \frac{1}{y^2}}, & A'B' &= \sqrt{\frac{1}{x^2} + \frac{1}{y^2}}. \end{aligned}$$

According to Exercise 12.25,

$$\begin{aligned} \text{ch } c &= \frac{AB \cdot A'B' + AB' \cdot A'B}{AA' \cdot BB'} = \\ &= \frac{\sqrt{x^2 + y^2} \cdot \sqrt{\frac{1}{x^2} + \frac{1}{y^2}} + \sqrt{\frac{1}{x^2} + y^2} \cdot \sqrt{x^2 + \frac{1}{y^2}}}{\left(\frac{1}{y} - y\right) \cdot \left(\frac{1}{x} - x\right)} = \\ \textcircled{8} \quad &= \frac{x^2 + y^2 + 1 + x^2 \cdot y^2}{(1 - y^2) \cdot (1 - x^2)} \\ &= \frac{1 + x^2}{1 - x^2} \cdot \frac{1 + y^2}{1 - y^2}. \end{aligned}$$

Finally note that $\textcircled{7}$ and $\textcircled{8}$ imply $\textcircled{6}$. □

Chapter 14

Affine geometry

Affine transformations

Affine geometry studies the so called *incidence structure* of the Euclidean plane. The incidence structure says which points lie on which lines and nothing else; we cannot talk about distances, angle measures and so on.

In other words, affine geometry studies the properties of the Euclidean plane which preserved under *affine transformations* defined below.

A bijection of Euclidean plane to itself is called *affine transformation* if it maps any line to a line.

Recall that three points are *collinear* if they lie on one line. Note that affine transformation sends collinear points to collinear; the following exercise gives a converse.

14.1. Exercise. *Assume f is a bijection from Euclidean plane to itself which sends collinear points to collinear points. Show that f is an affine transformation. (In other words, show that f maps noncollinear points to noncollinear.)*

14.2. Exercise. *Show that affine transformation sends a pair of parallel lines to a pair of parallel lines.*

Constructions

Let us consider geometric constructions with a ruler and a *parallel tool*; the latter makes possible to draw a line thru a given point parallel to a given line. By Exercisers 14.2, any construction with these two tools are invariant with respect to affine transformation. For example, to solve

the following exercise, it is sufficient to prove that midpoint of a given segment can be constructed with a ruler and a parallel tool.

14.3. Exercise. *Let M be the midpoint of segment $[AB]$ in the Euclidean plane. Assume that an affine transformation sends the points A , B and M to A' , B' and M' correspondingly. Show that M' is the midpoint of $[A'B']$.*

The following exercise will be used in the proof of Theorem 14.7.

14.4. Exercise. *Assume that the points with the coordinates $(0, 0)$, $(1, 0)$, $(a, 0)$ and $(b, 0)$ are given. Using a ruler and a parallel tool, construct the points with the coordinates $(a \cdot b, 0)$ and $(a + b, 0)$.*

14.5. Exercise. *Use ruler and parallel tool to construct the center of the given circle.*

Matrix form

Since the lines are defined in terms of metric; any motion of Euclidean plane is also an affine transformation.

Let us describe an example of an affine transformation which is not a motion.

Fix a coordinate system on the Euclidean plane. Let us use the column notation for the coordinates; that is, we will write $\begin{pmatrix} x \\ y \end{pmatrix}$ instead of (x, y) .

As it follows from the theorem below, the so called *shear mapping* $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x+k \cdot y \\ y \end{pmatrix}$ is an affine transformation. The shear mapping can change the angle between vertical and horizontal lines almost arbitrary. The latter can be used to prove impossibility of some constructions with a ruler and a parallel tool; here is one example.

14.6. Exercise. *Show that with a ruler and a parallel tool one cannot construct a line perpendicular to a given line.*

14.7. Theorem. *A map β from the plane to itself is an affine transformation if and only if*

$$\textcircled{1} \quad \beta: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} a \cdot x + b \cdot y + v \\ c \cdot x + d \cdot y + w \end{pmatrix}$$

for a fixed invertible matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and a vector $\begin{pmatrix} v \\ w \end{pmatrix}$.

In particular, any affine transformation of Euclidean plane is continuous.

In the proof of the “only if” part, we will use the following algebraic lemma:

14.8. Algebraic lemma. Assume $f: \mathbb{R} \rightarrow \mathbb{R}$ is a function such that for any $x, y \in \mathbb{R}$ we have

- (a) $f(1) = 1$,
- (b) $f(x + y) = f(x) + f(y)$,
- (c) $f(x \cdot y) = f(x) \cdot f(y)$.

Then f is the identity function; that is, $f(x) = x$ for any $x \in \mathbb{R}$.

Note that we do not assume that f is continuous.

The function f satisfying these three conditions is called *field automorphism*. Therefore, the lemma states that the identity function is the only automorphism of the field of real numbers. For the field of complex numbers, the conjugation $z \mapsto \bar{z}$ (see page 142) gives an example of nontrivial automorphism.

Proof. By (b) we have

$$f(0) + f(1) = f(0 + 1).$$

By (a)

$$f(0) + 1 = 1;$$

whence

$$\textcircled{2} \quad f(0) = 0.$$

Applying (b) again, we get that

$$0 = f(0) = f(x) + f(-x).$$

Therefore,

$$\textcircled{3} \quad f(-x) = -f(x) \quad \text{for any } x \in \mathbb{R}.$$

Applying (b) recurrently, we get that

$$\begin{aligned} f(2) &= f(1) + f(1) = 1 + 1 = 2; \\ f(3) &= f(2) + f(1) = 2 + 1 = 3; \\ &\dots \end{aligned}$$

Together with $\textcircled{3}$, the latter implies that

$$f(n) = n \quad \text{for any integer } n.$$

By (c)

$$f(m) = f\left(\frac{m}{n}\right) \cdot f(n).$$

Therefore

$$\textcircled{4} \quad f\left(\frac{m}{n}\right) = \frac{m}{n}$$

for any rational number $\frac{m}{n}$.

Assume $a \geq 0$. Then the equation $x \cdot x = a$ has a real solution $x = \sqrt{a}$. Therefore, $[f(\sqrt{a})]^2 = f(\sqrt{a}) \cdot f(\sqrt{a}) = f(a)$. Hence $f(a) \geq 0$. That is,

$$\textcircled{5} \quad a \geq 0 \implies f(a) \geq 0.$$

Applying $\textcircled{3}$, we also get

$$\textcircled{6} \quad a \leq 0 \implies f(a) \leq 0.$$

Now we are ready to the final step in the proof. Assume $f(a) \neq a$ for some $a \in \mathbb{R}$. Then there is a rational number $\frac{m}{n}$ which lies between a and $f(a)$; that is, the numbers

$$x = a - \frac{m}{n} \quad \text{and} \quad y = f(a) - \frac{m}{n}$$

have opposite signs.

By $\textcircled{4}$,

$$\begin{aligned} y + \frac{m}{n} &= f(a) = \\ &= f\left(x + \frac{m}{n}\right) = \\ &= f(x) + f\left(\frac{m}{n}\right) = \\ &= f(x) + \frac{m}{n}; \end{aligned}$$

that is, $f(x) = y$. By $\textcircled{5}$ and $\textcircled{6}$ the values x and y can not have opposite signs — a contradiction. $\square$

14.9. Lemma. Assume γ is an affine transformation which fix three points $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ on the coordinate plane. Then γ is the identity map; that is, $\gamma\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ for any point $\begin{pmatrix} x \\ y \end{pmatrix}$.

Proof. Since an affine transformation sends lines to lines, we get that each axes is mapped to itself.

According to Exercise 14.2, parallel lines are mapped to parallel lines. Therefore, we get that horizontal lines mapped to horizontal lines and vertical lines mapped to vertical. In other words,

$$\gamma\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f(x) \\ h(y) \end{pmatrix}.$$

for some functions $f, h: \mathbb{R} \rightarrow \mathbb{R}$.

Note that $f(1) = h(1) = 1$ and according to Exercise 14.4, both f and h satisfies the other two conditions of the algebraic lemma (14.8). Applying the lemma, we get that f and h are identity functions and so is γ . $\square$

Proof of Theorem 14.7; “if” part. Recall that matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible if

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a \cdot d - b \cdot c \neq 0;$$

in this case the matrix

$$\frac{1}{a \cdot d - b \cdot c} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

is the inverse of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Assume that the map β is described by ❶. Note that

$$\text{❷} \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \frac{1}{a \cdot d - b \cdot c} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \cdot \begin{pmatrix} x-v \\ y-w \end{pmatrix}.$$

is inverse of β . In particular, β is a bijection.

Any line in the plane is given by equation

$$\text{❸} \quad p \cdot x + q \cdot y + r = 0,$$

where $p \neq 0$ or $q \neq 0$ (see Exercise 7.23). Let us find $\begin{pmatrix} x \\ y \end{pmatrix}$ from its β -image by formula ❷ and substitute the result in ❸. We will get the equation of the image of the line. The equation has the same type as ❸, with different constants; in particular, it describes a line.

Therefore we proved that β is an affine transformation.

“Only if” part. Fix an affine transformation α . Set

$$\begin{aligned} \begin{pmatrix} v \\ w \end{pmatrix} &= \alpha \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\ \begin{pmatrix} a \\ c \end{pmatrix} &= \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \alpha \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\ \begin{pmatrix} b \\ d \end{pmatrix} &= \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \alpha \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned}$$

Note that the points $\alpha \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ do not lie on one line. Therefore, the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible.

For the affine transformation β defined by ❶ we have

$$\begin{aligned} \beta \begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \alpha \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\ \beta \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \alpha \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{aligned}$$

Consider the composition $\gamma = \alpha \circ \beta^{-1}$ is the identity map. Note that γ is an affine transformation which fix points $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. By Lemma 14.9, γ is the identity map or equivalently $\alpha = \beta$. $\square$

On inversive transformations

Recall that inversive plane is Euclidean plane with added a point at infinity, denoted by ∞ . We assume that every line passes thru ∞ . Recall that the term *circline* stands for *circle or line*.

An *inversive transformation* is a bijection from inversive plane to itself that sends circlines to circlines. *Inversive geometry* studies the *circline incidence structure* of inversive plane; it says which points lie on which circlines.

14.10. Theorem. *A map from inversive plane to itself is an inversive transformation if and only if it can be presented as a composition of inversions and reflections.*

Exercise 18.15 gives another description of inversive transformations which use complex coordinates.

Proof. According to Theorem 10.7 any inversion is an inversive transformation. Therefore, the same holds for composition of inversions and reflection.

To prove the converse, fix an inversive transformation α .

Assume $\alpha(\infty) = \infty$. Recall that any circline passing thru ∞ is a line. It follows that α maps lines to lines; that is, it is an affine transformation.

Note that α is not an arbitrary affine transformation — it maps circles to circles.

Composing α with a reflection, say ρ_1 , we can assume that $\alpha' = \rho_1 \circ \alpha$ maps the unit circle with center at the origin to a concentric circle.

Composing the obtained map α' with a *homothety*

$$\chi: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} k \cdot x \\ k \cdot y \end{pmatrix},$$

we can assume that $\alpha'' = \chi \circ \alpha'$ sends the unit circle to itself.

Composing the obtained map α'' with a reflection ρ_2 in a line thru the origin, we can assume that $\alpha''' = \rho_2 \circ \alpha''$ maps the point $(1, 0)$ to itself.

By Exercise 14.5, α''' fixes the center of the circle; that is, it fixes the origin.

The obtained map α''' is an affine transformation. Applying Theorem 14.7, together with the properties of α'' described above we get that

$$\alpha''': \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

for an invertible matrix $\begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix}$. Since the point $(0, 1)$ maps to the unit circle we get that

$$b^2 + d^2 = 1.$$

Since the point $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ maps to the unit circle we get that

$$(b + d)^2 = 1.$$

It follows

$$\alpha''': \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix};$$

that is, either α''' is the identity map or the reflection in the x -axis.

Note that the homothety χ is a composition of two inversions in concentric circles. Therefore, α is a composition of inversions and reflections if and only are so is α' , α'' and α''' .

In the remaining case $\alpha(\infty) \neq \infty$, set $P = \alpha(\infty)$. Consider an inversion β in a circle with center at P . Note that $\beta(P) = \infty$; therefore, $\beta \circ \alpha(\infty) = \infty$. Since β is inversive, so is $\beta \circ \alpha$. From above we get that $\beta \circ \alpha$ is a composition of reflections and inversions; therefore, so is α . $\square$

14.11. Exercise. *Show that inversive transformations preserve the angle between arcs up to sign.*

More precisely, assume $A'B'_1C'_1$, $A'B'_2C'_2$ are the images of two arcs AB_1C_1 , AB_2C_2 under an inversive transformation. Let α and α' denote the angle between the tangent half-lines to AB_1C_1 and AB_2C_2 at A and the angle between the tangent half-lines to $A'B'_1C'_1$ and $A'B'_2C'_2$ at A' correspondingly. Then

$$\alpha' = \pm\alpha.$$

14.12. Exercise. *Show that any reflection can be presented as a composition of three inversions.*

The exercise above together with Theorem 14.10, implies that any inversive map is a composition of inversions — no reflections needed.

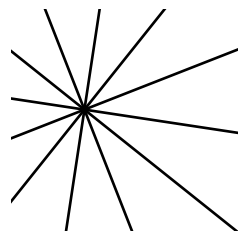
Chapter 15

Projective geometry

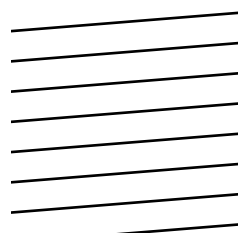
Real projective plane

In the Euclidean plane, two distinct lines might have one or zero points of intersection (in the latter case the lines are parallel). Our aim is to extend Euclidean plane by ideal points so that any two distinct lines will have exactly one point of intersection.

A collection of lines in the Euclidean plane is called *concurrent* if they all intersect at a single point or all of them pairwise parallel. A maximal set of concurrent lines in the plane is called *pencil*. There are two types of pencils: *central pencils* contain all lines passing thru a fixed point called the *center of the pencil* and *parallel pencil* contain pairwise parallel lines.



Each point in the Euclidean plane uniquely defines a central pencil with the center in it. Note that any two lines completely determine the pencil containing both.



Let us add one *ideal point* for each parallel pencil, and assume that all these ideal points lie on one *ideal line*. We also assume that the ideal line belongs to each parallel pencil.

We obtain the so called *real projective plane*. It comes with an incidence structure — we say that three points lie on one line if the corresponding pencils contain a common line.

Projective geometry studies this incidence structure on the projective plane. Loosely speaking, any statement in projective geometry can be formulated using only terms *collinear points*, *concurrent lines*.

Euclidean space

Let us repeat the construction of metric d_2 (page 11) in the space.

Let $\mathbb{R}^3$ denotes the set of all triples (x, y, z) of real numbers. Assume $A = (x_A, y_A, z_A)$ and $B = (x_B, y_B, z_B)$ are arbitrary points in $\mathbb{R}^3$. Define the metric on $\mathbb{R}^3$ the following way:

$$AB := \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2 + (z_A - z_B)^2}.$$

The obtained metric space is called *Euclidean space*.

Assume at least one of the real numbers a , b or c is distinct from zero. Then the subset of points $(x, y, z) \in \mathbb{R}^3$ described by equation

$$a \cdot x + b \cdot y + c \cdot z + d = 0$$

is called *plane*; here d is a real number.

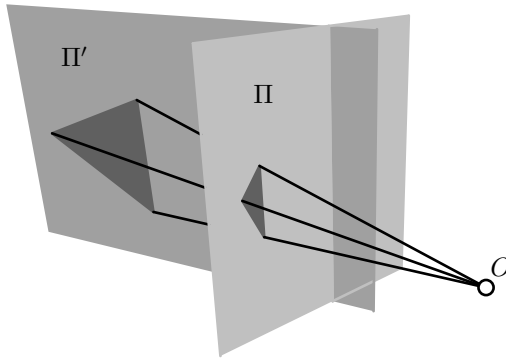
It is straightforward to show the following:

- ◊ Any plane in the Euclidean space is isometric to the Euclidean plane.
- ◊ Any three points in the space lie on a plane.
- ◊ An intersection of two distinct planes (if it is nonempty) is a line in each of these planes.

These statements make possible to generalize many notions and results from Euclidean plane geometry to the Euclidean space by applying plane geometry in the planes of the space.

Perspective projection

Consider two planes Π and Π' in the Euclidean space. Let O be a point which does not belong neither to Π nor Π' .



Let us define the *perspective projection from Π to Π' with center at O* . The projection of a point $P \in \Pi$ is defined as the intersection point $P' = \Pi' \cap (OP)$.

Note that the perspective projection sends collinear points to collinear. Indeed, assume three points P, Q, R lie on one line ℓ in Π and P', Q', R' are their images in Π' . Let Θ be the plane containing O and ℓ . Then all the points P, Q, R, P', Q', R' lie on Θ . Therefore, the points P', Q', R' lie on the intersection line $\ell' = \Theta \cap \Pi'$.

The perspective projection is not a bijection between the planes. Indeed, if the line (OP) is parallel to Π' (that is, if $(OP) \cap \Pi' = \emptyset$) then the perspective projection is not defined. Also, if $(OP') \parallel \Pi$ for $P' \in \Pi'$, then the point P' is not an image of the perspective projection.

Let us remind that a similar story happened with inversion. An inversion is not defined at its center; moreover, the center is not an inverse of any point. To deal with this problem we passed to inversive plane which is Euclidean plane extended by one ideal point.

A similar strategy works for perspective projection $\Pi \rightarrow \Pi'$, but this time real projective plane is the right choice of extension.

Let $\hat{\Pi}$ and $\hat{\Pi}'$ denote the corresponding real projective planes. Let us define a bijection between points in the real projective plane $\hat{\Pi}$ and the set Λ of all the lines passing thru O . If $P \in \Pi$, then take the line (OP) ; if P is an ideal point of $\hat{\Pi}$, so it is defined by a parallel pencil of lines, then take the line thru O parallel to the lines in this pencil.

The same construction gives a bijection between Λ and $\hat{\Pi}'$. Composing these two bijections $\hat{\Pi} \leftrightarrow \Lambda \leftrightarrow \hat{\Pi}'$, we get a bijection between $\hat{\Pi}$ and $\hat{\Pi}'$ which coincides with the perspective projection $P \mapsto P'$ where it is defined.

Note that the ideal line of $\hat{\Pi}$ maps to the intersection line of Π' and the plane thru O parallel to Π . Similarly the ideal line of $\hat{\Pi}'$ is the image of the intersection line of Π and the plane thru O parallel to Π' .

Strictly speaking we described a transformation from one real projective plane to another, but if we identify the two planes, say by fixing a coordinate system in each, we get a projective transformation from the plane to itself.

15.1. Exercise. Let O be the origin of (x, y, z) -coordinate space and the planes Π and Π' are given by the equations $x = 1$ and $y = 1$ correspondingly. The perspective projection from Π to Π' with center at O sends P to P' . Assume P has coordinates $(1, y, z)$, find the coordinates of P' .

For which points $P \in \Pi$ the perspective projection is undefined? Which points $P' \in \Pi'$ are not images of points under perspective projection?

Projective transformations

A bijection from the real projective plane to itself which sends lines to lines is called *projective transformation*.

Projective geometry studies the properties of real projective plane which preserved under projective transformations.

Note that any affine transformation defines a projective transformation on the corresponding real projective plane. We will call such projective transformations *affine*; these are projective transformations which send the ideal line to itself.

The perspective projection discussed in the previous section gives an example of projective transformation which is not affine.

15.2. Theorem. *Given a line ℓ in the real projective plane, there is a perspective projection which sends ℓ to the ideal line.*

Moreover, any projective transformation can be obtained as a composition of an affine transformation and a perspective projection.

Proof. Identify the projective plane with a plane Π in the space. Fix a point $O \notin \Pi$ and choose a plane Π' which is parallel to the plane containing ℓ and O . The corresponding perspective projection sends ℓ to the ideal line.

Assume α is a projective transformation.

If α sends ideal line to itself, then it has to be affine. It proves the theorem in this case.

If α sends the ideal line to the line ℓ , choose a perspective projection β which sends ℓ back to the ideal line. The composition $\beta \circ \alpha$ sends the ideal line to itself. That is, $\beta \circ \alpha$ is affine, hence the result. $\square$

Moving points to infinity

Theorem 15.2 makes possible to take any line in the projective plane and declare it to be ideal. In other words, we can choose preferred affine plane by removing one line from the projective plane. This construction provides a method for solving problems in projective geometry which will be illustrated by the following classical example:

15.3. Desargues' theorem. *Consider three concurrent lines (AA') , (BB') and (CC') in the real projective plane. Set*

$$X = (BC) \cap (B'C'), \quad Y = (CA) \cap (C'A'), \quad Z = (AB) \cap (A'B').$$

Then the points X , Y and Z are collinear.

Proof. Without loss of generality, we may assume that the line (XY) is ideal. If not, apply a perspective projection which sends the line (XY) to the ideal line.

That is, we can assume that

$$(BC) \parallel (B'C') \quad \text{and} \quad (CA) \parallel (C'A')$$

and we need to show that

$$(AB) \parallel (A'B').$$

Assume that the lines (AA') , (BB') and (CC') intersect at point O . Since $(BC) \parallel (B'C')$, the transversal property (7.6) implies that $\angle OBC = \angle OB'C'$ and $\angle OCB = \angle OC'B'$. By the AA similarity condition, $\triangle OBC \sim \triangle OB'C'$. In particular,

$$\frac{OB}{OB'} = \frac{OC}{OC'}.$$

The same way we get that $\triangle OAC \sim \triangle OA'C'$ and

$$\frac{OA}{OA'} = \frac{OC}{OC'}.$$

Therefore,

$$\frac{OA}{OA'} = \frac{OB}{OB'}.$$

By the SAS similarity condition, we get that $\triangle OAB \sim \triangle OA'B'$; in particular, $\angle OAB = \pm \angle OA'B'$.

Note that $\angle AOB = \angle A'OB'$. Therefore,

$$\angle OAB = \angle OA'B'.$$

By the transversal property 7.6, $(AB) \parallel (A'B')$.

The case $(AA') \parallel (BB') \parallel (CC')$ is done similarly. In this case the quadrilaterals $B'BCC'$ and $A'ACC'$ are parallelograms. Therefore,

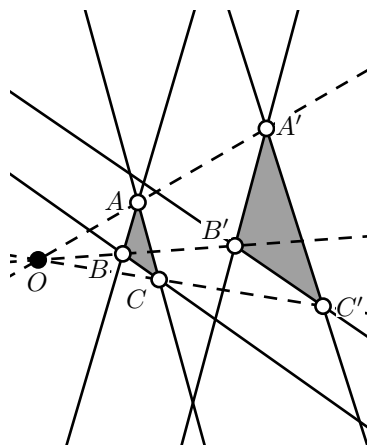
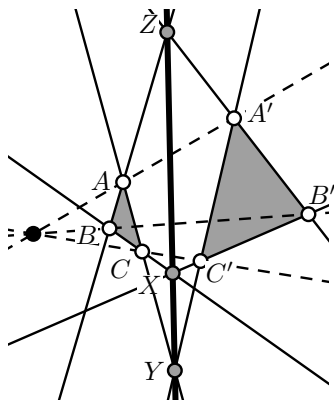
$$BB' = CC' = AA'.$$

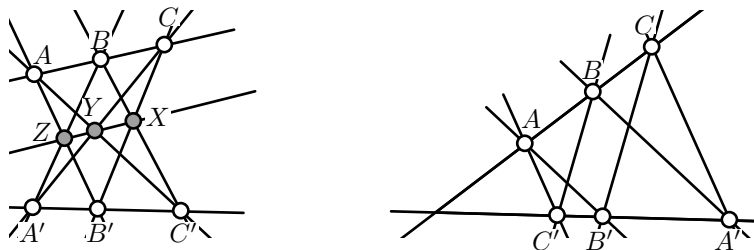
Hence $\square B'BAA'$ is a parallelogram and $(AB) \parallel (A'B')$. $\square$

Here is another classical theorem of projective geometry.

15.4. Pappus' theorem. Assume that two triples of points A, B, C , and A', B', C' are collinear. Set

$$X = (BC') \cap (B'C), \quad Y = (CA') \cap (C'A), \quad Z = (AB') \cap (A'B).$$





Then the points X, Y, Z are collinear.

Pappus' theorem can be proved the same way as Desargues' theorem.

Idea of the proof. Applying a perspective projection, we can assume that X and Y lie on the ideal line. It remains to show that Z lies on the ideal line.

In other words, assuming that $(AB') \parallel (A'B)$ and $(AC') \parallel (A'C)$, we need to show that $(BC') \parallel (B'C)$.

15.5. Exercise. Finish the proof of Pappus' theorem using the idea described above.

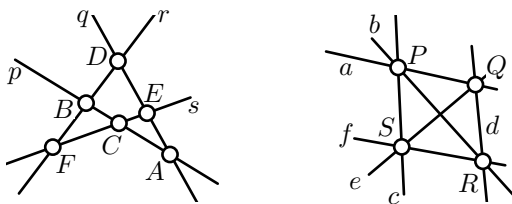
Duality

Assume that a bijection $P \leftrightarrow p$ between the set of lines and the set of points of a plane is given. That is, given a point P , we denote by p the corresponding line; and the other way around, given a line s we denote by S the corresponding point.

The bijection between points and lines is called *duality*¹ if

$$P \in s \iff p \ni S.$$

for any point P and line s .



Dual configurations.

¹The standard definition of duality is more general; we consider a special case which is also called *polarity*.

Existence of duality in a plane says that the lines and the points in this plane have the same rights in terms of incidence.

15.6. Exercise. *Show that Euclidean plane does not admit a duality.*

15.7. Theorem. *The real projective plane admits a duality.*

Proof. Consider a plane Π and a point $O \notin \Pi$ in the space; let $\hat{\Pi}$ denotes the corresponding real projective plane.

Recall that there is a natural bijection $\hat{\Pi} \leftrightarrow \Lambda$ between $\hat{\Pi}$ and the set Λ of all the lines passing thru O . Denote it by $P \leftrightarrow \dot{P}$; that is,

- ◇ if $P \in \Pi$, then $\dot{P} = (OP)$;
- ◇ if P is an ideal point of $\hat{\Pi}$, so P is defined as a parallel pencil of lines, set $\dot{P}$ to be the line thru O which is parallel to each lines in this pencil.

Similarly there is a natural bijection $s \leftrightarrow \dot{s}$ between lines in $\hat{\Pi}$ and all the planes passing thru O . If s is a line in Π , then $\dot{s}$ is the plane containing O and s ; if s is the ideal line of $\hat{\Pi}$, take $\dot{s}$ is the plane thru O parallel to Π .

It is straightforward to check that $\dot{P} \subset \dot{s}$ if and only if $P \in s$; that is, the bijections $P \leftrightarrow \dot{P}$ and $s \leftrightarrow \dot{s}$ remember all the incidence structure of the real projective plane $\hat{\Pi}$.

It remains to construct a bijection $\dot{s} \leftrightarrow \dot{\dot{S}}$ between the set of planes and the set of lines passing thru O such that

$$\textcircled{1} \quad \dot{r} \subset \dot{S} \quad \Longleftrightarrow \quad \dot{R} \supset \dot{s}$$

for any two lines $\dot{r}$ and $\dot{s}$ passing thru O .

Set $\dot{S}$ to be the plane thru O which is perpendicular to $\dot{s}$. Note that both conditions $\textcircled{1}$ are equivalent to $\dot{r} \perp \dot{s}$; hence the result follows. $\square$

15.8. Exercise. *Consider the Euclidean plane with (x, y) -coordinates; let O denotes the origin. Given a point $P \neq O$ with coordinates (a, b) consider the line p given by the equation $a \cdot x + b \cdot y = 1$.*

Show that the correspondence P to p extends to a duality of the real projective plane.

Which line corresponds to O ?

Which point of the real projective plane corresponds to the line $a \cdot x + b \cdot y = 0$?

The existence of duality in the real projective planes makes possible to formulate an equivalent dual statement to any statement in projective geometry. For example, the dual statement for “the points X, Y and Z lie on one line ℓ ” would be the “lines x, y and z intersect at one point L ”. Let us formulate the dual statement for Desargues’ theorem 15.3.

15.9. Dual Desargues' theorem. *Consider the collinear points X , Y and Z . Assume that*

$$X = (BC) \cap (B'C'), \quad Y = (CA) \cap (C'A'), \quad Z = (AB) \cap (A'B').$$

Then the lines (AA') , (BB') and (CC') are concurrent.

In this theorem the points X , Y and Z are dual to the lines (AA') , (BB') and (CC') in the original formulation, and the other way around.

Once Desargues' theorem is proved, applying duality (15.7) we get the dual Desargues' theorem. Note that the dual Desargues' theorem is the converse to the original Desargues' theorem 15.3.

15.10. Exercise. *Formulate the dual Pappus' theorem (see 15.4).*

15.11. Exercise.

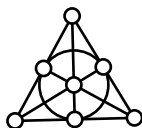
- (a) *Given two parallel lines, construct with a ruler only a third parallel line thru a given point.*
- (b) *Given a parallelogram, construct with a ruler only a line parallel to a given line thru a given point.*

Axioms

Note that the real projective plane described above satisfies the following set of axioms:

- p-I. Any two distinct points lie on a unique line.
- p-II. Any two distinct lines pass thru a unique point.
- p-III. There exist at least four points of which no three are collinear.

Let us take these three axioms as a definition of the *projective plane*; so the real projective plane discussed above becomes a particular example of projective plane.



There is an example of projective plane which contains exactly 3 points on each line. This is the so called *Fano plane* which you can see on the diagram; it contains 7 points and 7 lines. This is an example of *finite projective plane*; that is, projective plane with finitely many points.

15.12. Exercise. *Show that any line in projective plane contains at least three points.*

Consider the following analog of Axiom p-III:

p-III'. There exist at least four lines of which no three are concurrent.

15.13. Exercise. *Show that Axiom p-III' is equivalent to Axiom p-III. That is,*

p-I, p-II and p-III imply p-III',

and

p-I, p-II and p-III' imply p-III.

The exercise above shows that in the axiomatic system of projective plane, lines and points have the same rights. In fact, one can switch everywhere words “point” with “line”, “pass thru” with “lies on”, “collinear” with “concurrent” and we get an equivalent set of axioms — Axioms p-I and p-II convert into each other, and the same happens with the pair p-III and p-III'.

15.14. Exercise. *Assume that one of the lines in a finite projective plane contains exactly $n + 1$ points.*

- (a) *Show that each line contains exactly $n + 1$ points.*
- (b) *Show that the number of the points in the plane has to be*

$$n^2 + n + 1.$$

- (c) *Show that there is no projective plane with exactly 10 points.*
- (d) *Show that in any finite projective plane the number of points coincides with the number of lines.*

The number n in the above exercise is called *order* of finite projective plane. For example Fano plane has order 2. Here is one of the most famous open problem in finite geometry.

15.15. Conjecture. *The order of any finite projective plane is a power of a prime number.*

Chapter 16

Spherical geometry

Spherical geometry studies the surface of a unit sphere. This geometry has applications in cartography, navigation and astronomy.

The spherical geometry is a close relative of the Euclidean and hyperbolic geometries. Most of the theorems of hyperbolic geometry have spherical analogs, but spherical geometry is easier to visualize.

Euclidean space

Recall that Euclidean space is the set $\mathbb{R}^3$ of all triples (x, y, z) of real numbers such that the distance between a pair of points $A = (x_A, y_A, z_A)$ and $B = (x_B, y_B, z_B)$ is defined by the following formula:

$$AB := \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2 + (z_A - z_B)^2}.$$

The planes in the space are defined as the set of solutions of equation

$$a \cdot x + b \cdot y + c \cdot z + d = 0$$

for real numbers a, b, c and d such that at least one of the numbers a, b or c is not zero. Any plane in the Euclidean space is isometric to the Euclidean plane.

A sphere in the space is the direct analog of circle in the plane. Formally, *sphere* with center O and radius r is the set of points in the space that lie on the distance r from O .

Let A and B be two points on the unit sphere centered at O . The *spherical distance* from A to B (briefly AB_s) is defined as $|\angle AOB|$.

In spherical geometry, the role of lines play the *great circles*; that is, the intersection of the sphere with a plane passing thru O .

Note that the great circles do not form lines in the sense of Definition 1.8. Also, any two distinct great circles intersect at two antipodal points. In particular, the sphere does not satisfy the axioms of the neutral plane.

Pythagorean theorem

Here is an analog of the Pythagorean theorems (6.4 and 13.13) in spherical geometry.

16.1. Spherical Pythagorean Theorem. *Let $\triangle_s ABC$ be a spherical triangle with a right angle at C . Set $a = BC_s$, $b = CA_s$ and $c = AB_s$. Then*

$$\cos c = \cos a \cdot \cos b.$$

In the proof, we will use the notion of the scalar product which we are about to discuss.

Let $v_A = (x_A, y_A, z_A)$ and $v_B = (x_B, y_B, z_B)$ denote the position vectors of points A and B . The *scalar product* of the two vectors v_A and v_B in $\mathbb{R}^3$ is defined as

$$\textcircled{1} \quad \langle v_A, v_B \rangle := x_A \cdot x_B + y_A \cdot y_B + z_A \cdot z_B.$$

Assume both vectors v_A and v_B are nonzero; let φ denotes the angle measure between them. Then the scalar product can be expressed the following way:

$$\textcircled{2} \quad \langle v_A, v_B \rangle = |v_A| \cdot |v_B| \cdot \cos \varphi,$$

where

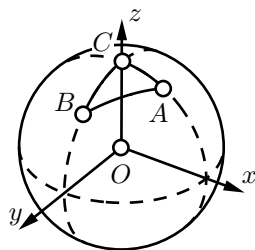
$$|v_A| = \sqrt{x_A^2 + y_A^2 + z_A^2}, \quad |v_B| = \sqrt{x_B^2 + y_B^2 + z_B^2}.$$

Now, assume that the points A and B lie on the unit sphere Σ in $\mathbb{R}^3$ centered at the origin. In this case $|v_A| = |v_B| = 1$. By $\textcircled{2}$ we get that

$$\textcircled{3} \quad \cos AB_s = \langle v_A, v_B \rangle.$$

Proof of the spherical Pythagorean Theorem.

Since the angle at C is right, we can choose the coordinates in $\mathbb{R}^3$ so that $v_C = (0, 0, 1)$, v_A lies in the xz -plane, so $v_A = (x_A, 0, z_A)$, and v_B lies in the yz -plane, so $v_B = (0, y_B, z_B)$.



Applying, ❸, we get that

$$\begin{aligned} z_A &= \langle v_C, v_A \rangle = \cos b, \\ z_B &= \langle v_C, v_B \rangle = \cos a. \end{aligned}$$

Applying, ❶ and ❸, we get that

$$\begin{aligned} \cos c &= \langle v_A, v_B \rangle = \\ &= x_A \cdot 0 + 0 \cdot y_B + z_A \cdot z_B = \\ &= \cos b \cdot \cos a. \end{aligned}$$

□

16.2. Exercise. Show that if $\triangle_s ABC$ is a spherical triangle with a right angle at C , and $AC_s = BC_s = \frac{\pi}{4}$, then $AB_s = \frac{\pi}{3}$.

Inversion of the space

The inversion in a sphere is defined the same way as we define the inversion in a circle.

Formally, let Σ be the sphere with the center O and radius r . The *inversion* in Σ of a point P is the point $P' \in [OP)$ such that

$$OP \cdot OP' = r^2.$$

In this case, the sphere Σ will be called the *sphere of inversion* and its center is called the *center of inversion*.

We also add ∞ to the space and assume that the center of inversion is mapped to ∞ and the other way around. The space $\mathbb{R}^3$ with the point ∞ will be called *inversive space*.

The inversion of the space has many properties of the inversion of the plane. Most important for us are the analogs of theorems 10.6, 10.7, 10.25 which can be summarized as follows:

16.3. Theorem. *The inversion in the sphere has the following properties:*

- (a) *Inversion maps a sphere or a plane into a sphere or a plane.*
- (b) *Inversion maps a circle or a line into a circle or a line.*
- (c) *Inversion preserves the cross-ratio; that is, if A' , B' , C' and D' are the inverses of the points A , B , C and D correspondingly, then*

$$\frac{AB \cdot CD}{BC \cdot DA} = \frac{A'B' \cdot C'D'}{B'C' \cdot D'A'}.$$

- (d) *Inversion maps arcs into arcs.*

(e) *Inversion preserves the absolute value of the angle measure between tangent half-lines to the arcs.*

We do not present the proofs here, but they nearly repeat the corresponding proofs in plane geometry. To prove (a), you will need in addition the following lemma; its proof is left to the reader.

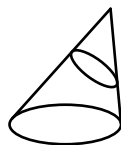
16.4. Lemma. *Let Σ be a subset of the Euclidean space which contains at least two points. Fix a point O in the space.*

Then Σ is a sphere if and only if for any plane Π passing thru O , the intersection $\Pi \cap \Sigma$ is either empty set, one point set or a circle.

The following observation helps to reduce part (b) to part (a).

16.5. Observation. *Any circle in the space is an intersection of two spheres.*

Let us define a *circular cone* as a set formed by line segments from a fixed point, called the *tip* of the cone, to all the points on a fixed circle, called the *base* of the cone; we always assume that the base does not lie in the same plane as the tip. We say that the cone is *right* if the center of the base circle is the foot point of the tip on the base plane; otherwise we call it *oblique*.



16.6. Exercise. *Let K be an oblique circular cone. Show that there is a plane Π that is not parallel to the base plane of K such that the intersection $\Pi \cap K$ is a circle.*

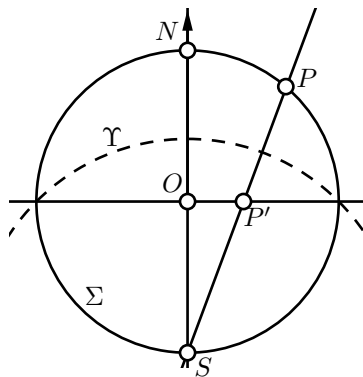
Stereographic projection

Consider the unit sphere Σ centered at the origin $(0, 0, 0)$. This sphere can be described by the equation $x^2 + y^2 + z^2 = 1$.

Let Π denotes the xy -plane; it is defined by the equation $z = 0$. Clearly, Π runs thru the center of Σ .

Let $N = (0, 0, 1)$ and $S = (0, 0, -1)$ denote the “north” and “south” poles of Σ ; these are the points on the sphere which have extremal distances to Π . Let Ω denotes the “equator” of Σ ; it is the intersection $\Sigma \cap \Pi$.

For any point $P \neq S$ on Σ , consider the line (SP) in the space. This line intersects Π in exactly one point, denoted by P' . Set $S' = \infty$.



The plane thru P , O and S .

The map $\xi_s: P \mapsto P'$ is called the *stereographic projection from Σ to Π with respect to the south pole*. The inverse of this map $\xi_s^{-1}: P' \mapsto P$ is called the *stereographic projection from Π to Σ with respect to the south pole*.

The same way, one can define the *stereographic projections ξ_n and ξ_n^{-1} with respect to the north pole N* .

Note that $P = P'$ if and only if $P \in \Omega$.

Note that if Σ and Π are as above, then the composition of the stereographic projections $\xi_s: \Sigma \rightarrow \Pi$ and $\xi_s^{-1}: \Pi \rightarrow \Sigma$ are the restrictions of the inversion in the sphere Υ with the center S and radius $\sqrt{2}$ to Σ and Π correspondingly.

From above and Theorem 16.3, it follows that the stereographic projection preserves the angles between arcs; more precisely *the absolute value of the angle measure* between arcs on the sphere.

This makes it particularly useful in cartography. A map of a big region of earth cannot be done in a constant scale, but using a stereographic projection, one can keep the angles between roads the same as on earth.

In the following exercises, we assume that Σ , Π , Υ , Ω , O , S and N are as above.

16.7. Exercise. Show that $\xi_n \circ \xi_s^{-1}$, the composition of stereographic projections from Π to Σ from S , and from Σ to Π from N is the inverse of the plane Π in Ω .

16.8. Exercise. Show that a stereographic projection $\Sigma \rightarrow \Pi$ sends the great circles to circlines on the plane which intersects Ω at two opposite points.

The following exercise is analogous to Lemma 13.10.

16.9. Exercise. Fix a point $P \in \Pi$ and let Q be another point in Π . Let P' and Q' denote their stereographic projections to Σ . Set $x = PQ$ and $y = P'Q'$. Show that

$$\lim_{x \rightarrow 0} \frac{y}{x} = \frac{2}{1 + OP^2}.$$

Central projection

The central projection is analogous to the projective model of hyperbolic plane which is discussed in Chapter 17.

Let Σ be the unit sphere centered at the origin which will be denoted by O . Let Π^+ denotes the plane defined by the equation $z = 1$. This plane is parallel to the xy -plane and it passes thru the north pole $N = (0, 0, 1)$ of Σ .

Recall that the northern hemisphere of Σ , is the subset of points $(x, y, z) \in \Sigma$ such that $z > 0$. The northern hemisphere will be denoted by Σ^+ .

Given a point $P \in \Sigma^+$, consider the half-line $[OP)$. Let P' denotes the intersection of $[OP)$ and Π^+ . Note that if $P = (x, y, z)$, then $P' = (\frac{x}{z}, \frac{y}{z}, 1)$. It follows that $P \leftrightarrow P'$ is a bijection between Σ^+ and Π^+ .

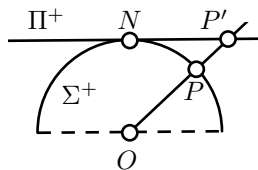
The described bijection $\Sigma^+ \leftrightarrow \Pi^+$ is called the *central projection* of the hemisphere Σ^+ .

Note that the central projection sends the intersections of the great circles with Σ^+ to the lines in Π^+ . The latter follows since the great circles are intersections of Σ with planes passing thru the origin as well as the lines in Π^+ are the intersection of Π^+ with these planes.

The following exercise is analogous to Exercise 17.4 in hyperbolic geometry.

16.10. Exercise. Let $\triangle_s ABC$ be a nondegenerate spherical triangle. Assume that the plane Π^+ is parallel to the plane passing thru A , B and C . Let A' , B' and C' denote the central projections of A , B and C .

- (a) Show that the midpoints of $[A'B']$, $[B'C']$ and $[C'A']$ are central projections of the midpoints of $[AB]_s$, $[BC]_s$ and $[CA]_s$ correspondingly.
- (b) Use part (a) to show that the medians of a spherical triangle intersect at one point.



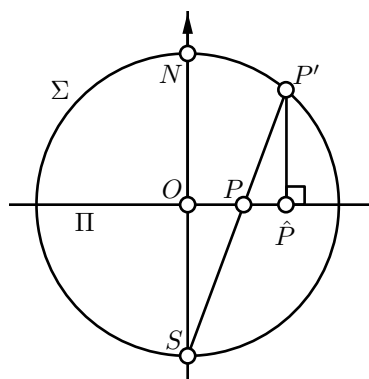
Chapter 17

Projective model

The *projective model* is another model of hyperbolic plane discovered by Beltrami; it is often called *Klein model*. The projective and conformal models are saying exactly the same thing but in two different languages. Some problems in hyperbolic geometry admit simpler proof using the projective model and others have simpler proof in the conformal model. Therefore, it is worth to know both.

Special bijection on the h-plane

Consider the conformal disc model with the absolute at the unit circle Ω centered at O . Choose a coordinate system (x, y) on the plane with the origin at O , so the circle Ω is described by the equation $x^2 + y^2 = 1$.



The plane thru P , O and S .

Let us think that our plane is the coordinate xy -plane in the Euclidean space; denote it by Π . Let Σ be the unit sphere centered at O ; it is described by the equation

$$x^2 + y^2 + z^2 = 1.$$

Set $S = (0, 0, -1)$ and $N = (0, 0, 1)$; these are the south and north poles of Σ .

Consider stereographic projection $\Pi \rightarrow \Sigma$ from S ; given point $P \in \Pi$ denote its image in Σ . Note that the h-plane is mapped to the *north hemisphere*; that is, to the set of points

(x, y, z) in Σ described by the inequality $z > 0$.

For a point $P' \in \Sigma$ consider its foot point $\hat{P}$ on Π ; this is the closest point to P' .

Note that the composition $P \leftrightarrow P' \leftrightarrow \hat{P}$ of these two maps gives a bijection from the h-plane to itself. Further note that $P = \hat{P}$ if and only if $P \in \Omega$ or $P = O$.

17.1. Exercise. Show that the bijection $P \leftrightarrow \hat{P}$ described above can be described the following way: set $\hat{O} = O$ and for any other point P take $\hat{P} \in [OP)$ such that

$$O\hat{P} = \frac{2 \cdot x}{1 + x^2},$$

where $x = OP$.

17.2. Lemma. Let $(PQ)_h$ be an h-line with the ideal points A and B . Then $\hat{P}, \hat{Q} \in [AB]$.

Moreover,

$$\textcircled{1} \quad \frac{A\hat{Q} \cdot B\hat{P}}{\hat{Q}B \cdot \hat{P}A} = \left(\frac{AQ \cdot BP}{QB \cdot PA} \right)^2.$$

In particular, if A, P, Q, B appear in the same order, then

$$PQ_h = \frac{1}{2} \cdot \ln \frac{A\hat{Q} \cdot B\hat{P}}{\hat{Q}B \cdot \hat{P}A}.$$

Proof. Consider the stereographic projection $\Pi \rightarrow \Sigma$ from the south pole S . Note that it fixes A and B ; denote by P' and Q' the images of P and Q ;

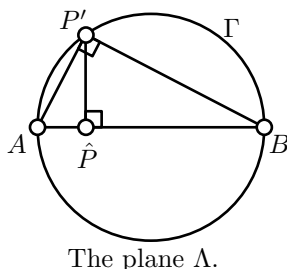
According to Theorem 16.3c,

$$\textcircled{2} \quad \frac{AQ \cdot BP}{QB \cdot PA} = \frac{AQ' \cdot BP'}{Q'B \cdot P'A}.$$

By Theorem 16.3e, each circline in Π which is perpendicular to Ω is mapped to a circle in Σ which is still perpendicular to Ω . It follows that the stereographic projection sends $(PQ)_h$ to the intersection of the north hemisphere of Σ with a plane perpendicular to Π .

Let Λ denotes the plane; it contains the points $A, B, P', \hat{P}$ and the circle $\Gamma = \Sigma \cap \Lambda$. (It also contains Q' and $\hat{Q}$ but we will not use these points for a while.)

Note that



- ◇ $A, B, P' \in \Gamma$,
- ◇ $[AB]$ is a diameter of Γ ,
- ◇ $(AB) = \Pi \cap \Lambda$,
- ◇ $\hat{P} \in [AB]$
- ◇ $(P'\hat{P}) \perp (AB)$.

Since $[AB]$ is the diameter of Γ , by Corollary 9.6, the angle $AP'B$ is right. Hence $\triangle A\hat{P}P' \sim \triangle AP'B \sim \triangle P'\hat{P}B$. In particular

$$\frac{AP'}{BP'} = \frac{A\hat{P}}{P'\hat{P}} = \frac{P'\hat{P}}{B\hat{P}}.$$

Therefore

$$\textcircled{3} \quad \frac{A\hat{P}}{B\hat{P}} = \left(\frac{AP'}{BP'} \right)^2.$$

The same way we get that

$$\textcircled{4} \quad \frac{A\hat{Q}}{B\hat{Q}} = \left(\frac{AQ'}{BQ'} \right)^2.$$

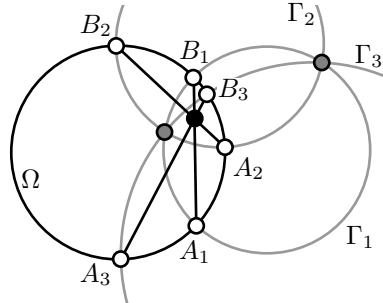
Finally, note that $\textcircled{2} + \textcircled{3} + \textcircled{4}$ imply $\textcircled{1}$.

The last statement follows from $\textcircled{1}$ and the definition of h-distance. Indeed,

$$\begin{aligned} PQ_h &:= \ln \frac{AQ \cdot BP}{QB \cdot PA} = \\ &= \ln \left(\frac{A\hat{Q} \cdot B\hat{P}}{\hat{Q}B \cdot \hat{P}A} \right)^{\frac{1}{2}} = \\ &= \frac{1}{2} \cdot \ln \frac{A\hat{Q} \cdot B\hat{P}}{\hat{Q}B \cdot \hat{P}A}. \end{aligned}$$

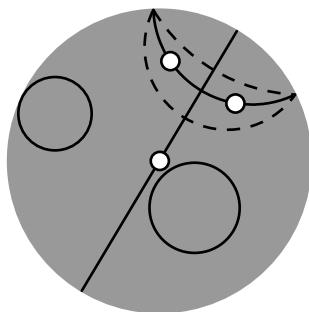
□

17.3. Exercise. Let Γ_1, Γ_2 and Γ_3 be three circles perpendicular to the circle Ω . Let $[A_1B_1]$, $[A_2B_2]$ and $[A_3B_3]$ denote the common chords of Ω and $\Gamma_1, \Gamma_2, \Gamma_3$ correspondingly. Show that the chords $[A_1B_1]$, $[A_2B_2]$ and $[A_3B_3]$ intersect at one point inside Ω if and only if Γ_1, Γ_2 and Γ_3 intersect at two points.

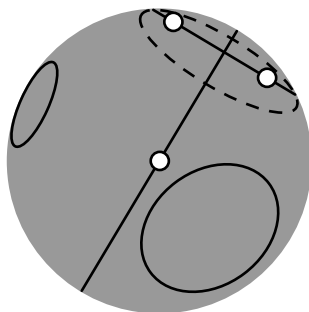


Projective model

The following picture illustrates the map $P \mapsto \hat{P}$ described in the previous section — if you take the picture on the left and apply the map $P \mapsto \hat{P}$,



Conformal model



Projective model

you get the picture on the right. The pictures are *conformal* and *projective model* of the hyperbolic plane correspondingly. The map $P \mapsto \hat{P}$ is a “translation” from one to another.

In the projective model things look different; some become simpler, other things become more complicated.

Lines. The h-lines in the projective model are chords of the absolute; more precisely, chords without its endpoints.

Circles and equidistants. The h-circles and equidistants in the projective model are certain type of ellipses and their open arcs.

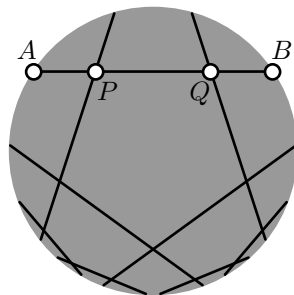
It follows since the stereographic projection sends circles on the plane to circles on the unit sphere and the foot point projection of circle back to the plane is an ellipse. (One may define *ellipse* as a foot point projection of a circle.)

Distance. Consider a pair of h-points P and Q . Let A and B be the ideal point of the h-line in projective model; that is, A and B are the intersections of the Euclidean line (PQ) with the absolute.

Then by Lemma 17.2,

$$\textcircled{5} \quad PQ_h = \frac{1}{2} \cdot \ln \frac{AQ \cdot BP}{QB \cdot PA},$$

assuming the points A, P, Q, B appear on the line in the same order.



Angles. The angle measures in the projective model are very different from the Euclidean angles and it is hard to figure out by looking on the picture. For example all the intersecting h-lines on the picture above are perpendicular. There are two useful exceptions:

- ◊ If O is the center of the absolute, then

$$\angle_h AOB = \angle AOB.$$

- ◊ If O is the center of the absolute and $\angle OAB = \pm \frac{\pi}{2}$, then

$$\angle_h OAB = \angle OAB = \pm \frac{\pi}{2}.$$

To find the angle measure in the projective model, you may apply a motion of the h-plane which moves the vertex of the angle to the center of the absolute; once it is done the hyperbolic and Euclidean angles have the same measure.

Motions. The motions of the h-plane in the conformal and projective models are relevant to inversive transformations and projective transformation in the same way. Namely:

- ◊ Inversive transformations that preserve the h-plane describe motions of the h-plane in the conformal model.
- ◊ Projective transformations that preserve h-plane describe motions of the h-plane in the projective model.¹

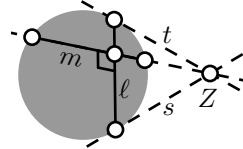
The following exercise is a hyperbolic analog of Exercise 16.10. This is the first example of a statement which admits an easier proof using the projective model.

17.4. Exercise. *Let P and Q be the points in h-plane which lie on the same distance from the center of the absolute. Observe that in the projective model, h -midpoint of $[PQ]_h$ coincides with the Euclidean midpoint of $[PQ]_h$.*

Conclude that if an h-triangle is inscribed in an h-circle, then its medians meet at one point.

Recall that an h-triangle might be also inscribed in a horocycle or an equidistant. Think how to prove the statement in this case.

17.5. Exercise. *Let ℓ and m are h-lines in the projective model. Let s and t denote the Euclidean lines tangent to the absolute at the ideal points of ℓ . Show that if the lines s , t and the extension of m intersect at one point, then ℓ and m are perpendicular h-lines.*



¹The idea described in the solution of Exercise 16.6 and in the sketch of proof of Theorem 19.13 can be used to construct many projective transformations of this type.

17.6. Exercise. Use the projective model to derive the formula for angle of parallelism (Proposition 13.1).

17.7. Exercise. Use projective model to find the inradius of the ideal triangle.

The projective model of h-plane can be used to give another proof of the hyperbolic Pythagorean theorem (13.13).

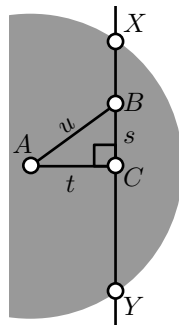
First let us recall its statement:

$$\textcircled{6} \quad \text{ch } c = \text{ch } a \cdot \text{ch } b,$$

where $a = BC_h$, $b = CA_h$ and $c = AB_h$ and $\triangle_h ACB$ is an h-triangle with right angle at C .

Note that we can assume that A is the center of the absolute. Set $s = BC$, $t = CA$, $u = AB$. According to the Euclidean Pythagorean theorem (6.4), we have

$$\textcircled{7} \quad u^2 = s^2 + t^2.$$



It remains to express a , b and c using s , u and t and show that $\textcircled{7}$ implies $\textcircled{6}$.

17.8. Advanced exercise. Finish the proof of hyperbolic Pythagorean theorem (13.13) indicated above.

Bolyai's construction

Assume we need to construct a line thru P asymptotically parallel to the given line ℓ in the h-plane.

If A and B are ideal points of ℓ in the projective model, then we could simply draw the Euclidean line (PA) . However the ideal points do not lie in the h-plane, therefore there is no way to use them in the construction.

In the following construction we assume that you know a compass-and-ruler construction of the perpendicular line; see Exercise 5.21.

17.9. Bolyai's construction.

1. Drop a perpendicular from P to ℓ ; denote it by m . Let Q be the foot point of P on ℓ .
2. Erect a perpendicular from P to m ; denote it by n .
3. Mark by R a point on ℓ distinct from Q .
4. Drop a perpendicular from R to n ; denote it by k .
5. Draw the circle Γ with center P and the radius QR . Mark by T a point of intersection of Γ with k .
6. The line $(PT)_h$ is asymptotically parallel to ℓ .

Chapter 18

Complex coordinates

In this chapter, we give an interpretation of inversive geometry using complex coordinates. The results of this chapter will not be used in this book, but they lead to deeper understanding of both concepts.

Complex numbers

Informally, a complex number is a number that can be put in the form

❶
$$z = x + i \cdot y,$$

where x and y are real numbers and $i^2 = -1$.

The set of complex numbers will be further denoted by $\mathbb{C}$. If x , y and z are as in ❶, then x is called the *real part* and y the *imaginary part* of the complex number z . Briefly it is written as

$$x = \operatorname{Re} z \quad \text{and} \quad y = \operatorname{Im} z.$$

On the more formal level, a complex number is a pair of real numbers (x, y) with the addition and multiplication described below. The formula $x + i \cdot y$ is only a convenient way to write the pair (x, y) .

❷
$$\begin{aligned} (x_1 + i \cdot y_1) + (x_2 + i \cdot y_2) &:= (x_1 + x_2) + i \cdot (y_1 + y_2); \\ (x_1 + i \cdot y_1) \cdot (x_2 + i \cdot y_2) &:= (x_1 \cdot x_2 - y_1 \cdot y_2) + i \cdot (x_1 \cdot y_2 + y_1 \cdot x_2). \end{aligned}$$

Complex coordinates

Recall that one can think of the Euclidean plane as the set of all pairs of real numbers (x, y) equipped with the metric

$$AB = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2},$$

where $A = (x_A, y_A)$ and $B = (x_B, y_B)$.

One can pack the coordinates (x, y) of a point in one complex number $z = x + i \cdot y$. This way we get a one-to-one correspondence between points of the Euclidean plane and $\mathbb{C}$. Given a point $Z = (x, y)$, the complex number $z = x + i \cdot y$ is called the *complex coordinate* of Z .

Note that if O , E and I are points in the plane with complex coordinates 0 , 1 and i , then $\angle EOI = \pm \frac{\pi}{2}$. Further, we assume that $\angle EOI = \frac{\pi}{2}$; if not, one has to change the direction of the y -coordinate.

Conjugation and absolute value

Let z be a complex number with real part x and imaginary part y . If $y = 0$, we say that the complex number z is *real* and if $x = 0$ we say that z is *imaginary*. The set of points with real (imaginary) complex coordinates is a line in the plane, which is called *real* (correspondingly *imaginary*) line. The real line will be denoted as $\mathbb{R}$.

The complex number $\bar{z} := x - i \cdot y$ is called the *complex conjugate* of z .

Let Z and $\bar{Z}$ be the points in the plane with the complex coordinates z and $\bar{z}$ correspondingly. Note that the point $\bar{Z}$ is the reflection of Z in the real line.

It is straightforward to check that

$$\textcircled{3} \quad x = \operatorname{Re} z = \frac{z + \bar{z}}{2}, \quad y = \operatorname{Im} z = \frac{z - \bar{z}}{i \cdot 2}, \quad x^2 + y^2 = z \cdot \bar{z}.$$

The last formula in $\textcircled{3}$ makes it possible to express the quotient $\frac{w}{z}$ of two complex numbers w and $z = x + i \cdot y$:

$$\frac{w}{z} = \frac{1}{z \cdot \bar{z}} \cdot w \cdot \bar{z} = \frac{1}{x^2 + y^2} \cdot w \cdot \bar{z}.$$

Note that

$$\overline{z + w} = \bar{z} + \bar{w}, \quad \overline{z - w} = \bar{z} - \bar{w}, \quad \overline{z \cdot w} = \bar{z} \cdot \bar{w}, \quad \overline{z/w} = \bar{z}/\bar{w}.$$

That is, the complex conjugation *respects* all the arithmetic operations.

The value

$$|z| := \sqrt{x^2 + y^2} = \sqrt{(x + i \cdot y) \cdot (x - i \cdot y)} = \sqrt{z \cdot \bar{z}}$$

is called the *absolute value* of z . If $|z| = 1$, then z is called a *unit complex number*.

18.1. Exercise. Show that $|v \cdot w| = |v| \cdot |w|$ for any $v, w \in \mathbb{C}$.

Note that if Z and W are points in the Euclidean plane, z and w are their complex coordinates, then

$$ZW = |z - w|.$$

The triangle inequality for the points with complex coordinates 0 , v and $v + w$ implies that

$$|v + w| \leq |v| + |w|$$

for any $v, w \in \mathbb{C}$; this inequality is also called *triangle inequality*.

18.2. Exercise. Use the identity

$$u \cdot (v - w) + v \cdot (w - u) + w \cdot (u - v) = 0$$

for $u, v, w \in \mathbb{C}$ and the triangle inequality to prove the Ptolemy's inequality (6.8).

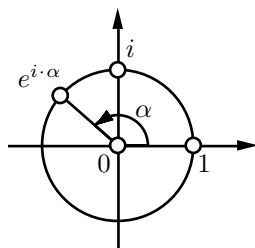
Euler's formula

Let α be a real number. The following identity is called *Euler's formula*:

④
$$e^{i \cdot \alpha} = \cos \alpha + i \cdot \sin \alpha.$$

In particular, $e^{i \cdot \pi} = -1$ and $e^{i \cdot \frac{\pi}{2}} = i$.

Geometrically, Euler's formula means the following: Assume that O and E are the points with complex coordinates 0 and 1 correspondingly. Assume



$$OZ = 1 \quad \text{and} \quad \angle EOZ \equiv \alpha,$$

then $e^{i \cdot \alpha}$ is the complex coordinate of Z . In particular, the complex coordinate of any point on the unit circle centered at O can be uniquely expressed as $e^{i \cdot \alpha}$ for some $\alpha \in (-\pi, \pi]$.

Why should you think that ④ is true? The proof of Euler's identity depends on the way you define the exponential function. If you never had to apply the exponential function to an imaginary number, you may take the right hand side in ④ as the definition of the $e^{i \cdot \alpha}$.

In this case, formally nothing has to be proved, but it is better to check that $e^{i \cdot \alpha}$ satisfies familiar identities. Mainly,

$$e^{i \cdot \alpha} \cdot e^{i \cdot \beta} = e^{i \cdot (\alpha + \beta)}.$$

The latter can be proved using ❷ and the following trigonometric formulas, which we assume to be known:

$$\begin{aligned}\cos(\alpha + \beta) &= \cos \alpha \cdot \cos \beta - \sin \alpha \cdot \sin \beta, \\ \sin(\alpha + \beta) &= \sin \alpha \cdot \cos \beta + \cos \alpha \cdot \sin \beta.\end{aligned}$$

If you know the power series for the sine, cosine and exponential function, the following might convince that the identity ❹ holds:

$$\begin{aligned}e^{i \cdot \alpha} &= 1 + i \cdot \alpha + \frac{(i \cdot \alpha)^2}{2!} + \frac{(i \cdot \alpha)^3}{3!} + \frac{(i \cdot \alpha)^4}{4!} + \frac{(i \cdot \alpha)^5}{5!} + \cdots = \\ &= 1 + i \cdot \alpha - \frac{\alpha^2}{2!} - i \cdot \frac{\alpha^3}{3!} + \frac{\alpha^4}{4!} + i \cdot \frac{\alpha^5}{5!} - \cdots = \\ &= \left(1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \cdots\right) + i \cdot \left(\alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - \cdots\right) = \\ &= \cos \alpha + i \cdot \sin \alpha.\end{aligned}$$

Argument and polar coordinates

As before, we assume that O and E are the points with complex coordinates 0 and 1 correspondingly.

Let Z be the a point distinct form O . Set $\rho = OZ$ and $\theta = \angle EOZ$. The pair (ρ, θ) is called the *polar coordinates* of Z .

If z is the complex coordinate of Z , then $\rho = |z|$. The value θ is called the *argument* of z (briefly, $\theta = \arg z$). In this case,

$$z = \rho \cdot e^{i \cdot \theta} = \rho \cdot (\cos \theta + i \cdot \sin \theta).$$

Note that

$$\arg(z \cdot w) \equiv \arg z + \arg w$$

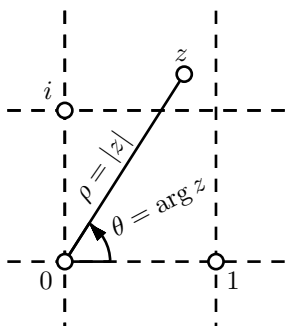
and

$$\arg \frac{z}{w} \equiv \arg z - \arg w$$

if $z \neq 0$ and $w \neq 0$. In particular, if Z, V, W are points with complex coordinates z, v and w correspondingly, then

❺

$$\begin{aligned}\angle VZW &= \arg \left(\frac{w - z}{v - z} \right) \equiv \\ &\equiv \arg(w - z) - \arg(v - z)\end{aligned}$$



if $\angle VZW$ is defined.

18.3. Exercise. Use the formula ⑤ to show that

$$\angle ZVW + \angle VWZ + \angle WZV \equiv \pi$$

for any $\triangle ZVW$ in the Euclidean plane.

18.4. Exercise. Assume that points V , W and Z have complex coordinates v , w and $z = v \cdot w$ correspondingly and the point O and E are as above. Show that

$$\triangle OEV \sim \triangle OWZ.$$

The following theorem is a reformulation of Theorem 9.10 which uses complex coordinates.

18.5. Theorem. Let $\square UVWZ$ be a quadrilateral and u , v , w and z be the complex coordinates of its vertices. Then $\square UVWZ$ is inscribed if and only if the number

$$\frac{(v-u) \cdot (z-w)}{(v-w) \cdot (z-u)}$$

is real.

The value $\frac{(v-u) \cdot (w-z)}{(v-w) \cdot (z-u)}$ is called the *complex cross-ratio*; it will be discussed in more details below.

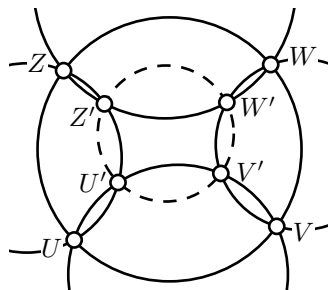
18.6. Exercise. Observe that the complex number $z \neq 0$ is real if and only if $\arg z = 0$ or π ; in other words, $2 \cdot \arg z \equiv 0$.

Use this observation to show that Theorem 18.5 is indeed a reformulation of Theorem 9.10.

18.7. Exercise. Let U , V , W , Z , U' , V' , W' and Z' be points on the plane with complex coordinates u , v , w , z , u' , v' , w' and z' correspondingly.

Assume that $\square UVWZ$, $\square UVV'U'$, $\square VWW'V'$, $\square WZZ'W'$ and $\square ZUU'Z'$ are inscribed.

- Express it using Theorem 18.5.
- Use (a) to show that $\square U'V'W'Z'$ is inscribed.



Method of complex coordinates

The following problem illustrates the method of complex coordinates.

18.8. Problem. Let $\triangle OPV$ and $\triangle OQW$ be isosceles right triangles such that

$$\angle VPO = \angle OQW = \frac{\pi}{2}$$

and M be the midpoint of $[VW]$. Assume P, Q and M are distinct points. Show that $\triangle PMQ$ is an isosceles right triangle.

Solution. Choose the complex coordinates so that O is the origin; denote by v, w, p, q, m the complex coordinates of the corresponding points.

Note that

$$v = (1 + i) \cdot p, \quad w = (1 - i) \cdot q.$$

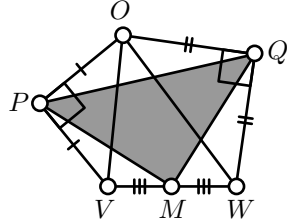
Therefore

$$\begin{aligned} m &= \frac{1}{2} \cdot (v + w) = \\ &= \frac{1+i}{2} \cdot p + \frac{1-i}{2} \cdot q. \end{aligned}$$

By a straightforward computations, we get that

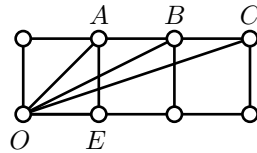
$$p - m = i \cdot (q - m).$$

In particular, $|p - m| = |q - m|$ and $\arg \frac{p-m}{q-m} = \frac{\pi}{2}$; that is, $PM = QM$ and $\angle QMP = \frac{\pi}{2}$. $\square$



18.9. Exercise. Consider three squares with common sides as on the diagram. Use the method of complex coordinates to show that

$$\angle EOA + \angle EOB + \angle EOC = \pm \frac{\pi}{2}.$$



Fractional linear transformations

18.10. Exercise. Watch video “Möbius transformations revealed” by Douglas Arnold and Jonathan Rogness. (It is available on YouTube.)

The complex plane $\mathbb{C}$ extended by one ideal number ∞ is called the *extended complex plane*. It is denoted by $\hat{\mathbb{C}}$, so $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

A *fractional linear transformation* or *Möbius transformation* of $\hat{\mathbb{C}}$ is a function of one complex variable z which can be written as

$$f(z) = \frac{a \cdot z + b}{c \cdot z + d},$$

where the coefficients a, b, c, d are complex numbers satisfying $a \cdot d - b \cdot c \neq 0$. (If $a \cdot d - b \cdot c = 0$ the function defined above is a constant and is not considered to be a fractional linear transformation.)

In case $c \neq 0$, we assume that

$$f(-d/c) = \infty \quad \text{and} \quad f(\infty) = a/c;$$

and if $c = 0$ we assume

$$f(\infty) = \infty.$$

Elementary transformations

The following three types of fractional linear transformations are called *elementary*:

1. $z \mapsto z + w$,
2. $z \mapsto w \cdot z$ for $w \neq 0$,
3. $z \mapsto \frac{1}{z}$.

The geometric interpretations. Let O denotes the point with the complex coordinate 0.

The first map $z \mapsto z + w$, corresponds to the so called *parallel translation* of the Euclidean plane, its geometric meaning should be evident.

The second map is called the *rotational homothety* with the center at O . That is, the point O maps to itself and any other point Z maps to a point Z' such that $OZ' = |w| \cdot OZ$ and $\angle ZOZ' = \arg w$.

The third map can be described as a composition of the inversion in the unit circle centered at O and the reflection in $\mathbb{R}$ (the composition can be taken in any order). Indeed, $\arg z \equiv -\arg \frac{1}{z}$. Therefore,

$$\arg z = \arg(1/\bar{z});$$

that is, if the points Z and Z' have complex coordinates z and $1/\bar{z}$, then $Z' \in [OZ)$. Clearly, $OZ = |z|$ and $OZ' = |1/\bar{z}| = \frac{1}{|z|}$. Therefore, Z' is the inverse of Z in the unit circle centered at O .

Finally, the reflection of Z' in $\mathbb{R}$, has complex coordinate $\frac{1}{z} = \overline{(1/\bar{z})}$.

18.11. Proposition. *The map $f: \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ is a fractional linear transformation if and only if it can be expressed as a composition of elementary transformations.*

Proof; the “only if” part. Fix a fractional linear transformation

$$f(z) = \frac{a \cdot z + b}{c \cdot z + d}.$$

Assume $c \neq 0$. Then

$$\begin{aligned} f(z) &= \frac{a}{c} - \frac{a \cdot d - b \cdot c}{c \cdot (c \cdot z + d)} = \\ &= \frac{a}{c} - \frac{a \cdot d - b \cdot c}{c^2} \cdot \frac{1}{z + \frac{d}{c}}. \end{aligned}$$

That is,

$$\textcircled{6} \quad f(z) = f_4 \circ f_3 \circ f_2 \circ f_1(z),$$

where f_1 , f_2 , f_3 and f_4 are elementary transformations of the following form:

$$\begin{aligned} \diamond f_1(z) &= z + \frac{d}{c}, \\ \diamond f_2(z) &= \frac{1}{z}, \\ \diamond f_3(z) &= -\frac{a \cdot d - b \cdot c}{c^2} \cdot z, \\ \diamond f_4(z) &= z + \frac{a}{c}. \end{aligned}$$

If $c = 0$, then

$$f(z) = \frac{a \cdot z + b}{d}.$$

In this case

$$f(z) = f_2 \circ f_1(z),$$

where $f_1(z) = \frac{a}{d} \cdot z$ and $f_2(z) = z + \frac{b}{d}$.

“If” part. We need to show that by composing elementary transformations, we can only get fractional linear transformations. Note that it is sufficient to check that the composition of a fractional linear transformations

$$f(z) = \frac{a \cdot z + b}{c \cdot z + d}.$$

with any elementary transformation $z \mapsto z + w$, $z \mapsto w \cdot z$ and $z \mapsto \frac{1}{z}$ is a fractional linear transformations.

The latter is done by means of direct calculations.

$$\begin{aligned} \frac{a \cdot (z + w) + b}{c \cdot (z + w) + d} &= \frac{a \cdot z + (b + a \cdot w)}{c \cdot z + (d + c \cdot w)}, \\ \frac{a \cdot (w \cdot z) + b}{c \cdot (w \cdot z) + d} &= \frac{(a \cdot w) \cdot z + b}{(c \cdot w) \cdot z + d}, \\ \frac{a \cdot \frac{1}{z} + b}{c \cdot \frac{1}{z} + d} &= \frac{b \cdot z + a}{d \cdot z + c}. \end{aligned}$$

□

18.12. Corollary. *The image of a circline under a fractional linear transformation is a circline.*

Proof. By Proposition 18.11, it is sufficient to check that each elementary transformation sends a circline to a circline.

For the first and second elementary transformation, the latter is evident.

As it was noted above, the map $z \mapsto \frac{1}{z}$ is a composition of inversion and reflection. By Theorem 10.11, the inversion sends a circline to a circline. Hence the result. $\square$

18.13. Exercise. Show that the inverse of a fractional linear transformation is a fractional linear transformation.

18.14. Exercise. Given distinct values $z_0, z_1, z_\infty \in \hat{\mathbb{C}}$, construct a fractional linear transformation f such that

$$f(z_0) = 0, \quad f(z_1) = 1 \quad \text{and} \quad f(z_\infty) = \infty.$$

Show that such a transformation is unique.

18.15. Exercise. Show that any inversion is a composition of the complex conjugation and a fractional linear transformation.

Use Theorem 14.10 to conclude that any inversive transformation is either fractional linear transformation or a complex conjugate to a fractional linear transformation.

Complex cross-ratio

Given four distinct complex numbers u, v, w and z , the complex number

$$\frac{(u-w) \cdot (v-z)}{(v-w) \cdot (u-z)}$$

is called the *complex cross-ratio*; it will be denoted by $(u, v; w, z)$.

If one of the numbers u, v, w, z is ∞ , then the complex cross-ratio has to be defined by taking the appropriate limit; in other words, we assume that $\frac{\infty}{\infty} = 1$. For example,

$$(u, v; w, \infty) = \frac{(u-w)}{(v-w)}.$$

Assume that U, V, W and Z are the points with complex coordinates u, v, w and z correspondingly. Note that

$$\begin{aligned} \frac{UW \cdot VZ}{VW \cdot UZ} &= |(u, v; w, z)|, \\ \angle WUZ + \angle ZVW &= \arg \frac{u-w}{u-z} + \arg \frac{v-z}{v-w} \equiv \\ &\equiv \arg(u, v; w, z). \end{aligned}$$

It makes it possible to reformulate Theorem 10.6 using the complex coordinates the following way:

18.16. Theorem. *Let $UWVZ$ and $U'W'V'Z'$ be two quadrilaterals such that the points U' , W' , V' and Z' are inverses of U , W , V , and Z correspondingly. Assume u , w , v , z , u' , w' , v' and z' are the complex coordinates of U , W , V , Z , U' , W' , V' and Z' correspondingly.*

Then

$$(u', v'; w', z') = \overline{(u, v; w, z)}.$$

The following exercise is a generalization of the theorem above. It admits a short and simple solution which uses Proposition 18.11.

18.17. Exercise. *Show that complex cross-ratios are invariant under fractional linear transformations.*

That is, if a fractional linear transformation maps four distinct complex numbers u, v, w, z to complex numbers u', v', w', z' respectively, then

$$(u', v'; w', z') = (u, v; w, z).$$

Schwarz–Pick theorem

The following theorem shows that the metric in the conformal disc model naturally appears in other branches of mathematics. We do not give a proof, but it can be found in any textbook on geometric complex analysis.

Let $\mathbb{D}$ denotes the unit disc in the complex plane centered at 0; that is, a complex number z belongs to $\mathbb{D}$ if and only if $|z| < 1$.

Let us use the disc $\mathbb{D}$ as a h -plane in the conformal disc model; the h -distance between $z, w \in \mathbb{D}$ will be denoted by $d_h(z, w)$.

A function $f: \mathbb{D} \rightarrow \mathbb{C}$ is called *holomorphic* if for every $z \in \mathbb{D}$ there is a complex number s such that

$$f(z + w) = f(z) + s \cdot w + o(|w|).$$

In other words, f is *complex-differentiable* at any $z \in \mathbb{D}$. The number s above is called the derivative of f at z and is denoted by $f'(z)$.

18.18. Schwarz–Pick theorem. *Assume $f: \mathbb{D} \rightarrow \mathbb{D}$ is a holomorphic function. Then*

$$d_h(f(z), f(w)) \leq d_h(z, w)$$

for any $z, w \in \mathbb{D}$.

If the equality holds for one pair of distinct numbers $z, w \in \mathbb{D}$, then it holds for any pair. In this case f is a fractional linear transformation as well as a motion of the h -plane.

18.19. Exercise. Show that if a fractional linear transformation f appears in the equality case of Schwarz–Pick theorem, then it can be written as

$$f(z) = \frac{v \cdot z + \bar{w}}{w \cdot z + \bar{v}},$$

where $|w| < |v|$.

18.20. Exercise. Show that

$$\operatorname{th}[\tfrac{1}{2} \cdot d_h(z, w)] = \left| \frac{z - w}{1 - z \cdot \bar{w}} \right|.$$

Conclude the inequality in Schwarz–Pick theorem can be rewritten as

$$\left| \frac{z' - w'}{1 - z' \cdot \bar{w}'} \right| \leq \left| \frac{z - w}{1 - z \cdot \bar{w}} \right|,$$

where $z' = f(z)$ and $w' = f(w)$.

18.21. Exercise. Show that the Schwarz lemma stated below follows from Schwarz–Pick theorem.

18.22. Schwarz lemma. Let $f: \mathbb{D} \rightarrow \mathbb{D}$ be a holomorphic function and $f(0) = 0$. Then $|f(z)| \leq |z|$ for any $z \in \mathbb{D}$.

Moreover, if equality holds for some $z \neq 0$, then there is a unit complex number u such that $f(z) = u \cdot z$ for any $z \in \mathbb{D}$.

Chapter 19

Geometric constructions

Geometric constructions have great pedagogical value as an introduction to mathematical proofs. We were using construction problems everywhere starting from Chapter 5.

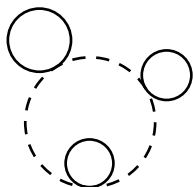
In this chapter we briefly discuss the classical results in geometric constructions.

Classical problems

In this section we list a couple of classical construction problems; each known for more than a thousand years. The solutions of the following two problems are quite nontrivial.

19.1. Problem of Brahmagupta. *Construct an inscribed quadrilateral with given sides.*

19.2. Problem of Apollonius. *Construct a circle which is tangent to three given circles.*



The following exercise is a simplified version of the problem of Apollonius, which is still nontrivial.

19.3. Exercise. *Construct a circle which passes thru a given point and is tangent to two intersecting lines.*

The following three problems cannot be solved in principle; that is, the needed compass-and-ruler construction does not exist.

Doubling the cube. *Construct the side of a new cube, which has the volume twice as big as the volume of a given cube.*

In other words, given a segment of the length a , one needs to construct a segment of length $\sqrt[3]{2} \cdot a$.

Squaring the circle. *Construct a square with the same area as a given circle.*

If r is the radius of the given circle, we need to construct a segment of length $\sqrt{\pi} \cdot r$.

Angle trisection. *Divide the given angle into three equal angles.*

In fact, there is no compass-and-ruler construction which trisects angle with measure $\frac{\pi}{3}$. Existence of such a construction would imply constructability of a regular 9-gon which is prohibited by the following famous result:

A regular n -gon inscribed in a circle with center O is a sequence of points $A_1 \dots A_n$ on the circle such that

$$\angle A_n O A_1 = \angle A_1 O A_2 = \dots = \angle A_{n-1} O A_n = \pm \frac{2}{n} \cdot \pi.$$

The points $A_1, \dots, A_n$ are *vertexes*, the segments $[A_1 A_2], \dots, [A_n A_1]$ are *sides* and the remaining segments $[A_i A_j]$ are *diagonals* of the n -gon.

A construction of a regular n -gon, therefore, is reduced to the construction of an angle with size $\frac{2}{n} \cdot \pi$.

19.4. Gauss–Wantzel theorem. *A regular n -gon can be constructed with a ruler and a compass if and only if n is the product of a power of 2 and any number of distinct Fermat primes.*

A *Fermat prime* is a prime number of the form $2^k + 1$ for some integer k . Only five Fermat primes are known today:

$$3, 5, 17, 257, 65537.$$

For example,

- ◊ one can construct a regular 340-gon since $340 = 2^2 \cdot 5 \cdot 17$ and 5 as well as 17 are Fermat primes;
- ◊ one cannot construct a regular 7-gon since 7 is not a Fermat prime;
- ◊ one cannot construct a regular 9-gon; altho $9 = 3 \cdot 3$ is a product of two Fermat primes, these primes are not distinct.

The impossibility of these constructions was proved only in 19th century. The method used in the proofs is indicated in the next section.

Constructible numbers

In the classical compass-and-ruler constructions initial configuration can be completely described by a finite number of points; each line is defined

by two points on it and each circle is described by its center and a point on it (equivalently, you may describe a circle by three points on it).

The same way the result of construction can be described by a finite collection of points.

We may always assume that the initial configuration has at least two point; if not add one or two points to the configuration. Moreover, applying a rescaling to whole plane, we can assume that the first two points in the initial configuration lie on distance 1 from each other.

In this case we can choose a coordinate system, such that one of the initial points is the origin $(0, 0)$ and yet another initial point has the coordinates $(1, 0)$. In this coordinate system, the initial configuration of n points is described by $2 \cdot n - 4$ numbers — their coordinates $x_3, y_3, \dots, x_n, y_n$.

It turns out that the coordinates of any point constructed with a compass and ruler can be written thru the numbers $x_3, y_3, \dots, x_n, y_n$ using the four arithmetic operations “+”, “−”, “·”, “/” and the square root “ $\sqrt{}$ ”.

For example, assume we want to find the points $X_1 = (x_1, y_1)$ and $X_2 = (x_2, y_2)$ of the intersections of a line passing thru $A = (x_A, y_A)$ and $B = (x_B, y_B)$ and the circle with center $O = (x_O, y_O)$ which passes thru the point $W = (x_W, y_W)$. Let us write the equations of the circle and the line in the coordinates (x, y) :

$$\begin{cases} (x - x_O)^2 + (y - y_O)^2 = (x_W - x_O)^2 + (y_W - y_O)^2, \\ (x - x_A) \cdot (y_B - y_A) = (y - y_A) \cdot (x_B - x_A). \end{cases}$$

Expressing y from the second equation and substituting the result in the first one, gives us a quadratic equation in x , which can be solved using “+”, “−”, “·”, “/” and “ $\sqrt{}$ ” only.

The same can be performed for the intersection of two circles. The intersection of two lines is even simpler; it is described as a solution of two linear equations and can be expressed using only four arithmetic operations; the square root “ $\sqrt{}$ ” is not needed.

On the other hand, it is easy to produce compass-and-ruler constructions which produce segments of the lengths $a + b$ and $a - b$ from two given segments of lengths $a > b$.

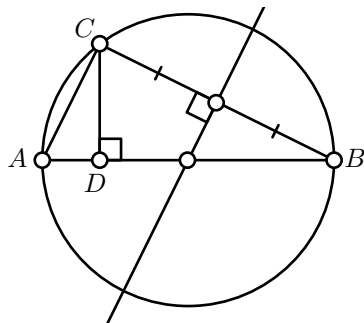
To perform “·”, “/” and “ $\sqrt{}$ ” consider the following diagram: let $[AB]$ be a diameter of a circle; fix a point C on the circle and let D be the foot point of C on $[AB]$. Note that

$$\triangle ABC \sim \triangle ACD \sim \triangle BDC.$$

It follows that $AD \cdot DC = BD^2$.

Using this diagram, one should guess the compass-and-ruler constructions which produce segments of lengths $\sqrt{a \cdot b}$ and $\frac{a^2}{b}$.

For example, to construct $\sqrt{a \cdot b}$, do the following: (1) construct points A , B and $D \in [AB]$ such that $AD = a$ and $BD = b$; (2) construct the circle Γ on the diameter $[AB]$; (3) draw the line ℓ thru D perpendicular to (AB) ; (4) let C be an intersection of Γ and ℓ . Then $DC = \sqrt{a \cdot b}$.



Taking 1 for a or b above, we can produce $\sqrt{a}$, a^2 , $\frac{1}{b}$. Combining these constructions we can produce $a \cdot b = (\sqrt{a \cdot b})^2$, $\frac{a}{b} = a \cdot \frac{1}{b}$. In other words we produced a *compass-and-ruler calculator*, which can do “+”, “−”, “.”, “/” and “ $\sqrt{}$ ”.

The discussion above gives a sketch of the proof of the following theorem:

19.5. Theorem. *Assume that the initial configuration of geometric construction is given by the points $A_1 = (0, 0)$, $A_2 = (1, 0)$, $A_3 = (x_3, y_3), \dots, A_n = (x_n, y_n)$. Then a point $X = (x, y)$ can be constructed using a compass-and-ruler construction if and only if both coordinates x and y can be expressed from the integer numbers and $x_3, y_3, x_4, y_4, \dots, x_n, y_n$ using the arithmetic operations “+”, “−”, “.”, “/” and the square root “ $\sqrt{}$ ”.*

The numbers which can be expressed from the given numbers using the arithmetic operations and the square root “ $\sqrt{}$ ” are called *constructible*; if the list of given numbers is not given, then we can only use the integers.

The theorem above translates any compass-and-ruler construction problem into a purely algebraic language. For example:

- ◊ The impossibility of a solution for doubling the cube problem states that $\sqrt[3]{2}$ is not a constructible number. That is $\sqrt[3]{2}$ cannot be expressed thru integers using “+”, “−”, “.”, “/” and “ $\sqrt{}$ ”.
- ◊ The impossibility of a solution for squaring the circle states that $\sqrt{\pi}$, or equivalently π , is not a constructible number.
- ◊ The Gauss–Wantzel theorem says for which integers n the number $\cos \frac{2\pi}{n}$ is constructible.

Some of these statements might look evident, but rigorous proofs require some knowledge of abstract algebra (namely, field theory) which is out of the scope of this book.

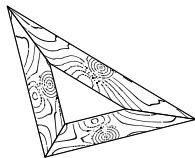
In the next section, we discuss similar but simpler examples of impossible constructions with an unusual tool.

19.6. Exercise.

- (a) Show that diagonal or regular pentagon is $\frac{1+\sqrt{5}}{2}$ times larger than its side.
- (b) Use (a) to make a compass-and-ruler construction of a regular pentagon.

Constructions with a set square

A set square is a construction tool shown on the picture — it can produce a line thru a given point which makes the angles $\frac{\pi}{2}$ or $\pm\frac{\pi}{4}$ to a given line and it can be also used as a ruler; that is, it can produce a line thru a given pair of points.

**19.7. Exercise.** *Trisect a given segment with a set square.*

Let us consider set-square constructions. Following the same lines as in the previous section, we can define *set-square constructible numbers* and prove the following analog of Theorem 19.5:

19.8. Theorem. *Assume that the initial configuration of a geometric construction is given by the points $A_1 = (0, 0)$, $A_2 = (1, 0)$, $A_3 = (x_3, y_3), \dots, A_n = (x_n, y_n)$. Then a point $X = (x, y)$ can be constructed using a set-square construction if and only if both coordinates x and y can be expressed from the integer numbers and $x_3, y_3, x_4, y_4, \dots, x_n, y_n$ using the arithmetic operations “+”, “−”, “.”, “/”.*

Let us apply this theorem to show the impossibility of some constructions with a set square.

Note that if all the coordinates $x_3, y_3, \dots, x_n, y_n$ are rational numbers, then the theorem above implies that with a set square, one can only construct the points with rational coordinates. A point with both rational coordinates is called *rational*, and if at least one of the coordinates is irrational, then the point is called *irrational*.

19.9. Exercise. *Show that an equilateral triangle in the Euclidean plane has at least one irrational point.*

Conclude that with a set square, one cannot construct an equilateral triangle.

19.10. Exercise. *Make a set-square construction which verifies if the given triangle is equilateral. (We assume that we can “verify” if two constructed points coincide.)*

More impossible constructions

In this section we discuss yet another source of impossible constructions.

Recall that a *circumtool* produces a circle passing thru any given three points or a line if all three points lie on one line. Let us restate Exercise 10.9.

Exercise. *Show that with a circumtool only, it is impossible to construct the center of a given circle Γ .*

Remark. In geometric constructions, we allow to choose some free points, say any point on the plane, or a point on a constructed line, or a point which does not lie on a constructed line and so on.

In principle, when you make such a free choice it is possible to mark the center of Γ by accident. Nevertheless, we do not accept such a coincidence as true construction; we say that a construction produces the center if it produces it for any free choices.

Solution. Arguing by contradiction, assume we have a construction of the center.

Apply an inversion in a circle perpendicular to Γ to the whole construction. According to Corollary 10.16, the circle Γ maps to itself. Since the inversion sends a circline to a circline, we get that the whole construction is mapped to an equivalent construction; that is, a construction with a different choice of free points.

According to Exercise 10.8, the inversion sends the center of Γ to another point. That is, following the same construction, we can end up at a different point — a contradiction. $\square$

19.11. Exercise. *Show that there is no circumtool-only construction which verifies if the given point is the center of a given circle. (We assume that we can only “verify” if two constructed points coincide.)*

A similar example of impossible constructions for a ruler and a parallel tool is given in Exercise 14.6.

Let us discuss yet another example for a ruler-only construction. Note that ruler-only constructions are invariant with respect to the projective transformations. In particular, to solve the following exercise, it is sufficient to construct a projective transformation which fixes two points A and B and moves its midpoint.

19.12. Exercise. *Show that the midpoint of a given segment cannot be constructed with only a ruler.*

The following theorem is a stronger version of the exercise above.

19.13. Theorem. *The center of a given circle cannot be constructed with only a ruler.*

Sketch of the proof. It is sufficient to construct a projective transformation which sends the given circle Γ to a circle Γ' such that the center of Γ' is not the image of the center of Γ . (Compare the construction with Exercise 16.6.)

Let Γ be a circle which lies in the plane Π in the Euclidean space.

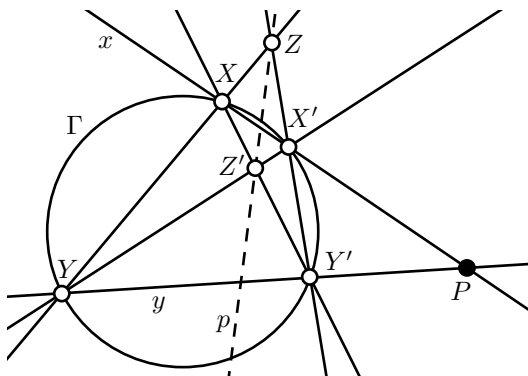
By Theorem 16.3, the inverse of a circle in a sphere is a circle or a line. Fix a sphere Σ with the center O so that the inversion Γ' of Γ is a circle and the plane Π' containing Γ' is not parallel to Π ; any sphere Σ in a general position will do.

Let Z and Z' denote the centers of Γ and Γ' . Note that $Z' \notin (OZ)$. It follows that the perspective projection $\Pi \rightarrow \Pi'$ with center at O sends Γ to Γ' , but Z' is not the image of Z . $\square$

Construction of a polar

In this section we describe a powerful trick that can be used in the constructions only with ruler.

Assume Γ is a circle in the plane and $P \notin \Gamma$. Draw two lines x and y thru P which intersect Γ at two pairs of points X, X' and Y, Y' . Let $Z = (XY) \cap (X'Y')$ and $Z' = (XY') \cap (X'Y)$. Consider the line $p = (ZZ')$.



The following claim will be used in the constructions without a proof.

19.14. Claim. *The constructed line $p = (ZZ')$ does not depend on the choice of the lines x and y . Moreover, $P \mapsto p$ is a duality (see page 124).*

The line p is called the polar of P with respect to Γ . The same way the point P is called the polar of the line p with respect to Γ .

19.15. Exercise. Let p be the polar line of point P with respect to the circle Γ . Assume that p intersects Γ at points V and W . Show that the lines (PV) and (PW) are tangent to Γ .

Come up with a ruler-only construction of the tangent lines to the given circle Γ thru the given point $P \notin \Gamma$.

19.16. Exercise. Assume two concentric circles Γ and Γ' are given. Construct the common center of Γ and Γ' with a ruler only.

Chapter 20

Area

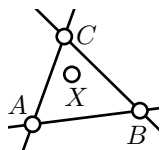
The area functional will be defined by Theorem 20.7. This theorem is given without proof, but it follows immediately from the properties of *Lebesgue measure* on the plane. The construction of Lebesgue measure typically use the method of coordinates and it is included in any textbook in real analysis. Based on this theorem, we develop the concept of area with no cheating.

We choose this approach since any rigorous introduction to area is tedious. We do not want to cheat and at the same time we do not want to waste your time; soon or later you will have to learn Lebesgue measure if it is not done already.

Solid triangles

We say that the point X lies *inside* a nondegenerate triangle ABC if the following three condition hold:

- ◊ A and X lie on the same side of the line (BC) ;
- ◊ B and X lie on the same side of the line (CA) ;
- ◊ C and X lie on the same side of the line (AB) .



The set of all points inside $\triangle ABC$ and on its sides $[AB]$, $[BC]$, $[CA]$ will be called *solid triangle ABC* and denoted by $\blacktriangle ABC$.

20.1. Exercise. Show that any solid triangle is convex; that is, if $X, Y \in \blacktriangle ABC$, then $[XY] \subset \blacktriangle ABC$.

The notations $\triangle ABC$ and $\blacktriangle ABC$ look similar, they also have close but different meanings, which better not to confuse. Recall that $\triangle ABC$ is an ordered triple of distinct points (see page 17), while $\blacktriangle ABC$ is an infinite set of points.

In particular, $\blacktriangle ABC = \blacktriangle BAC$ for any triangle ABC . Indeed, any point which belong to the set $\blacktriangle ABC$ also belongs to the set $\blacktriangle BAC$ and the other way around. On the other hand, $\triangle ABC \neq \triangle BAC$ simply because the ordered triple of points (A, B, C) is distinct from the ordered triple (B, A, C) .

Note that $\blacktriangle ABC \cong \blacktriangle BAC$ even if $\triangle ABC \not\cong \triangle BAC$, where congruence of the sets $\blacktriangle ABC$ and $\blacktriangle BAC$ is understood the following way:

20.2. Definition. Two sets $\mathcal{S}$ and $\mathcal{T}$ in the plane are called congruent (briefly $\mathcal{S} \cong \mathcal{T}$) if $\mathcal{T} = f(\mathcal{S})$ for some motion f of the plane.

If $\triangle ABC$ is not degenerate and

$$\blacktriangle ABC \cong \blacktriangle A'B'C',$$

then after relabeling the vertices of $\triangle ABC$ we will have

$$\triangle ABC \cong \triangle A'B'C'.$$

Indeed it is sufficient to show that if f is a motion which maps $\blacktriangle ABC$ to $\blacktriangle A'B'C'$, then f maps each vertex of $\triangle ABC$ to a vertex $\triangle A'B'C'$. The latter follows from the characterization of vertexes of solid triangles given in the following exercise:

20.3. Exercise. Let $\triangle ABC$ be nondegenerate and $X \in \blacktriangle ABC$. Show that X is a vertex of $\triangle ABC$ if and only if there is a line ℓ which intersects $\blacktriangle ABC$ at the single point X .

Polygonal sets

Elementary set on the plane is a set of one of the following three types:

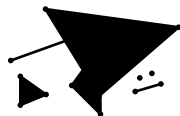
- ◊ one-point set;
- ◊ segment;
- ◊ solid triangle.

A set in the plane is called *polygonal* if it can be presented as a union of finite collection of elementary sets.

Note that according to this definition, the empty set $\emptyset$ is a polygonal set. Indeed, $\emptyset$ is a union of an empty collection of elementary sets.

A polygonal set is called *degenerate* if it can be presented as union of finite number of one-point sets and segments.

If X and Y lie on opposite sides of the line (AB) , then the union $\blacktriangle AXB \cup \blacktriangle BYA$ is a polygonal set which is called *solid quadrilateral* $AXBY$ and denoted by $\blacksquare AXBY$. In particular, we can talk about *solid parallelograms*, *rectangles* and *squares*.





Typically a polygonal set admits many presentation as union of a finite collection of elementary sets. For example, if $\square AXBY$ is a parallelogram, then

$$\blacksquare AXBY = \blacktriangle AXB \cup \blacktriangle BYA = \blacktriangle XAY \cup \blacktriangle YBX.$$

20.4. Exercise. Show that a solid square is not degenerate.

20.5. Exercise. Show that a circle is not a polygonal set.

Definition of area

20.6. Claim. For any two polygonal sets $\mathcal{P}$ and $\mathcal{Q}$, the union $\mathcal{P} \cup \mathcal{Q}$ as well as the intersection $\mathcal{P} \cap \mathcal{Q}$ are also polygonal sets.

A class of sets which closed with respect to union and intersection is called a *ring of sets*. The claim above, therefore, states that polygonal sets in the plane form a ring of sets.

Informal proof. Let us present $\mathcal{P}$ and $\mathcal{Q}$ as a union of finite collection of elementary sets $\mathcal{P}_1, \dots, \mathcal{P}_k$ and $\mathcal{Q}_1, \dots, \mathcal{Q}_n$ correspondingly.

Note that

$$\mathcal{P} \cup \mathcal{Q} = \mathcal{P}_1 \cup \dots \cup \mathcal{P}_k \cup \mathcal{Q}_1 \cup \dots \cup \mathcal{Q}_n.$$

Therefore, $\mathcal{P} \cup \mathcal{Q}$ is polygonal.

Note that $\mathcal{P} \cap \mathcal{Q}$ is the union of sets $\mathcal{P}_i \cap \mathcal{Q}_j$ for all i and j . Therefore, in order to show that $\mathcal{P} \cap \mathcal{Q}$ is polygonal, it is sufficient to show that each $\mathcal{P}_i \cap \mathcal{Q}_j$ is polygonal for any pair i, j .



The diagram should suggest an idea for the proof of the latter statement in case if $\mathcal{P}_i$ and $\mathcal{Q}_j$ are solid triangles. The other cases are simpler; a formal proof can be built on Exercise 20.1 $\square$

The following theorem defines the area as a function which returns a real number for any polygonal set and satisfying certain conditions. We omit the proof of this theorem. It follows from the construction of Lebesgue measure which can be found in any text book on real analysis.

20.7. Theorem. For each polygonal set $\mathcal{P}$ in the Euclidean plane there is a real number s called area of $\mathcal{P}$ (briefly $s = \text{area } \mathcal{P}$) such that

(a) $\text{area } \emptyset = 0$ and $\text{area } \mathcal{K} = 1$ where $\mathcal{K}$ a solid square with unit side;

(b) the conditions

$$\begin{aligned}\mathcal{P} \cong \mathcal{Q} &\Rightarrow \text{area } \mathcal{P} = \text{area } \mathcal{Q}; \\ \mathcal{P} \subset \mathcal{Q} &\Rightarrow \text{area } \mathcal{P} \leq \text{area } \mathcal{Q}; \\ \text{area } \mathcal{P} + \text{area } \mathcal{Q} &= \text{area}(\mathcal{P} \cup \mathcal{Q}) + \text{area}(\mathcal{P} \cap \mathcal{Q})\end{aligned}$$

hold for any two polygonal sets $\mathcal{P}$ and $\mathcal{Q}$.
Moreover, the area function

$$\mathcal{P} \mapsto \text{area } \mathcal{P}$$

is uniquely defined by the above conditions.

20.8. Proposition. For any polygonal set $\mathcal{P}$ in the Euclidean plane, we have

$$\text{area } \mathcal{P} \geq 0.$$

Proof. Since $\emptyset \subset \mathcal{P}$, we get that

$$\text{area } \emptyset \leq \text{area } \mathcal{P}.$$

Since $\text{area } \emptyset = 0$ the result follows. $\square$

Vanishing area and subdivisions

20.9. Proposition. Any one-point set as well as any segment in the Euclidean plane have vanishing area.

Proof. Fix a line segment $[AB]$. Consider a solid square $\blacksquare ABCD$.

Note that given a positive integer n , there are n disjoint segments $[A_1B_1], \dots, [A_nB_n]$ in $\blacksquare ABCD$, such that each $[A_iB_i]$ is congruent to $[AB]$ in the sense of the Definition 20.2.

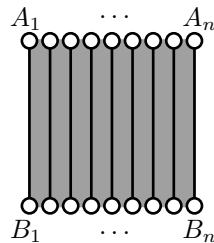
Applying the last identity in Theorem 20.7 few times, we get that

$$\begin{aligned}n \cdot \text{area}[AB] &= \text{area}([A_1B_1] \cup \dots \cup [A_nB_n]) \leq \\ &\leq \text{area}(\blacksquare ABCD)\end{aligned}$$

That is,

$$\text{area}[AB] \leq \frac{1}{n} \cdot \text{area}(\blacksquare ABCD)$$

for any positive integer n . Therefore, $\text{area}[AB] \leq 0$.



On the other hand, by Proposition 20.8,

$$\text{area}[AB] \geq 0$$

and hence the result.

For any one-point set $\{A\}$ we have that $\emptyset \subset \{A\} \subset [AB]$. Therefore,

$$0 \leq \text{area}\{A\} \leq \text{area}[AB] = 0.$$

Hence $\text{area}\{A\} = 0$. □

20.10. Corollary. *Any degenerate polygonal set in the Euclidean plane has vanishing area.*

Proof. Let $\mathcal{P}$ be a degenerate set, say

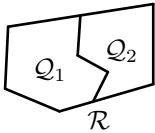
$$\mathcal{P} = [A_1 B_1] \cup \cdots \cup [A_n B_n] \cup \{C_1, \dots, C_k\}.$$

Applying Theorem 20.7 together with Proposition 20.8, we get that

$$\begin{aligned} \text{area } \mathcal{P} &\leq \text{area}[A_1 B_1] + \cdots + \text{area}[A_n B_n] + \\ &\quad + \text{area}\{C_1\} + \cdots + \text{area}\{C_k\}. \end{aligned}$$

By Proposition 20.9, the right hand side vanish.

On the other hand, $\text{area } \mathcal{P} \geq 0$ (see 20.8), hence the result. □



We say that polygonal set $\mathcal{P}$ is *subdivided* into two polygonal sets $\mathcal{Q}_1$ and $\mathcal{Q}_2$ if $\mathcal{P} = \mathcal{Q}_1 \cup \mathcal{Q}_2$ and the intersection $\mathcal{Q}_1 \cap \mathcal{Q}_2$ is degenerate. (Recall that according to Claim 20.6, the set $\mathcal{Q}_1 \cap \mathcal{Q}_2$ is polygonal.)

20.11. Proposition. *Assume polygonal sets $\mathcal{P}$ is subdivided into two polygonal set $\mathcal{Q}_1$ and $\mathcal{Q}_2$. Then*

$$\text{area } \mathcal{P} = \text{area } \mathcal{Q}_1 + \text{area } \mathcal{Q}_2.$$

Proof. By Theorem 20.7,

$$\text{area } \mathcal{P} = \text{area } \mathcal{Q}_1 + \text{area } \mathcal{Q}_2 - \text{area}(\mathcal{Q}_1 \cap \mathcal{Q}_2).$$

Since $\mathcal{Q}_1 \cap \mathcal{Q}_2$ is degenerate, by Corollary 20.10,

$$\text{area}(\mathcal{Q}_1 \cap \mathcal{Q}_2) = 0$$

and hence the result. □

Area of solid rectangles

20.12. Theorem. *The solid rectangle in the Euclidean plane with sides a and b has area $a \cdot b$.*

20.13. Algebraic lemma. *Assume that a function s returns a nonnegative real number $s(a, b)$ for any pair of positive real numbers (a, b) and it satisfies the following identities:*

$$\begin{aligned} s(1, 1) &= 1; \\ s(a, b + c) &= s(a, b) + s(a, c) \\ s(a + b, c) &= s(a, c) + s(b, c) \end{aligned}$$

for any $a, b, c > 0$. Then

$$s(a, b) = a \cdot b$$

for any $a, b > 0$.

The proof is similar to the proof of Lemma 14.8.

Proof. Note that if $a > a'$ and $b > b'$ then

$$\textcircled{1} \quad s(a, b) \geq s(a', b').$$

Indeed, since s returns nonnegative numbers, we get that

$$\begin{aligned} s(a, b) &= s(a', b) + s(a - a', b) \geq \\ &\geq s(a', b) = \\ &\geq s(a', b') + s(a', b - b') \geq \\ &\geq s(a', b'). \end{aligned}$$

Applying the second and third identity few times we get that

$$s(a, m \cdot b) = s(m \cdot a, b) = m \cdot s(a, b)$$

for any positive integer m . Therefore

$$\begin{aligned} s\left(\frac{k}{l}, \frac{m}{n}\right) &= k \cdot s\left(\frac{1}{l}, \frac{m}{n}\right) = \\ &= k \cdot m \cdot s\left(\frac{1}{l}, \frac{1}{n}\right) = \\ &= k \cdot m \cdot \frac{1}{l} \cdot s\left(1, \frac{1}{n}\right) = \\ &= k \cdot m \cdot \frac{1}{l} \cdot \frac{1}{n} \cdot s(1, 1) = \\ &= \frac{k}{l} \cdot \frac{m}{n} \end{aligned}$$

for any positive integers k, l, m and n . That is, the needed identity holds for any pair of rational numbers $a = \frac{k}{l}$ and $b = \frac{m}{n}$.

Arguing by contradiction, assume $s(a, b) \neq a \cdot b$ for some pair of positive real numbers (a, b) . We will consider two cases: $s(a, b) > a \cdot b$ and $s(a, b) < a \cdot b$.

If $s(a, b) > a \cdot b$, we can choose a positive integer n such that

$$\textcircled{2} \quad s(a, b) > (a + \frac{1}{n}) \cdot (b + \frac{1}{n}).$$

Set $k = \lfloor a \cdot n \rfloor + 1$ and $m = \lfloor b \cdot n \rfloor + 1$; equivalently, k and m are positive integers such that

$$a < \frac{k}{n} \leq a + \frac{1}{n} \quad \text{and} \quad b < \frac{m}{n} \leq b + \frac{1}{n}.$$

By $\textcircled{1}$, we get that

$$\begin{aligned} s(a, b) &\leq s(\frac{k}{n}, \frac{m}{n}) = \\ &= \frac{k}{n} \cdot \frac{m}{n} \leq \\ &\leq (a + \frac{1}{n}) \cdot (b + \frac{1}{n}), \end{aligned}$$

which contradicts $\textcircled{2}$.

The case $s(a, b) < a \cdot b$ is similar. Fix a positive integer n such that $a > \frac{1}{n}$, $b > \frac{1}{n}$ and

$$\textcircled{3} \quad s(a, b) < (a - \frac{1}{n}) \cdot (b - \frac{1}{n}).$$

Set $k = \lceil a \cdot n \rceil - 1$ and $m = \lceil b \cdot n \rceil - 1$; that is,

$$a > \frac{k}{n} \geq a - \frac{1}{n} \quad \text{and} \quad b > \frac{m}{n} \geq b - \frac{1}{n}.$$

Applying $\textcircled{1}$ again, we get that

$$\begin{aligned} s(a, b) &\geq s(\frac{k}{n}, \frac{m}{n}) = \\ &= \frac{k}{n} \cdot \frac{m}{n} \geq \\ &\geq (a - \frac{1}{n}) \cdot (b - \frac{1}{n}), \end{aligned}$$

which contradicts $\textcircled{3}$. $\square$

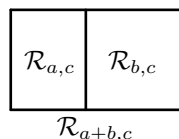
Proof of Theorem 20.12. Let $\mathcal{R}_{a,b}$ denotes the solid rectangle with sides a and b . Set

$$s(a, b) = \text{area } \mathcal{R}_{a,b}.$$

By theorem 20.7, $s(1, 1) = 1$. That is, the first identity in the algebraic lemma holds.

Note that the rectangle $\mathcal{R}_{a+b,c}$ can be subdivided into two rectangle congruent to $\mathcal{R}_{a,c}$ and $\mathcal{R}_{b,c}$. Therefore, by Proposition 20.11,

$$\text{area } \mathcal{R}_{a+b,c} = \text{area } \mathcal{R}_{a,c} + \text{area } \mathcal{R}_{b,c}$$



That is, the second identity in the algebraic lemma holds. The proof of the third identity is analogues.

It remains to apply the algebraic lemma. $\square$

Area of solid parallelograms

20.14. Proposition. *Let $\square ABCD$ be a parallelogram in the Euclidean plane, $a = AB$ and h be the distance between the lines (AB) and (CD) . Then*

$$\text{area}(\blacksquare ABCD) = a \cdot h.$$

Proof. Let A' and B' denote the foot points of A and B on the line (CD) .

Note that $ABB'A'$ is a rectangle with sides a and h . By Proposition 20.12,

$$\textcircled{4} \quad \text{area}(\blacksquare ABB'A') = h \cdot a.$$

Without loss of generality, we may assume that $\blacksquare ABCA'$ contains $\blacksquare ABCD$ and $\blacksquare ABB'A'$.

In this case $\blacksquare ABB'D$ admits two subdivisions. First into $\blacksquare ABCD$ and $\blacktriangle AA'D$. Second into $\blacksquare ABB'A'$ and $\blacktriangle BB'C$.

By Proposition 20.11,

$$\begin{aligned} \textcircled{5} \quad \text{area}(\blacksquare ABCD) + \text{area}(\blacktriangle AA'D) &= \\ &= \text{area}(\blacksquare ABB'A') + \text{area}(\blacktriangle BB'C). \end{aligned}$$

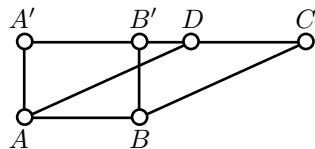
Note that

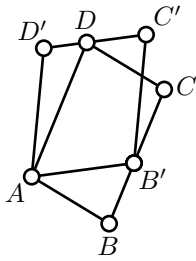
$$\textcircled{6} \quad \triangle AA'D \cong \triangle BB'C.$$

Indeed, since the quadrilaterals $ABB'A'$ and $ABCD$ are parallelograms, by Lemma 7.16, we have that $AA' = BB'$, $AD = BC$ and $DC = AB = A'B'$. It follows that $A'D = B'C$. Applying the SSS congruence condition, we get $\textcircled{6}$.

In particular,

$$\textcircled{7} \quad \text{area}(\blacktriangle BB'C) = \text{area}(\blacktriangle AA'D).$$





Subtracting ⑦ from ⑤, we get that

$$\text{area}(\blacksquare ABCD) = \text{area}(\blacksquare ABB'D').$$

From ④, the statement follows. $\square$

20.15. Exercise. Assume $\square ABCD$ and $\square AB'C'D'$ are two parallelograms such that $B' \in [BC]$ and $D \in [C'D']$. Show that

$$\text{area}(\blacksquare ABCD) = \text{area}(\blacksquare AB'C'D').$$

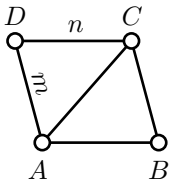
Area of solid triangles

20.16. Theorem. Let $a = BC$ and h_A to be the altitude from A in $\triangle ABC$. Then

$$\text{area}(\triangle ABC) = \frac{1}{2} \cdot a \cdot h_A.$$

Remark. It is acceptable to write $\text{area}(\triangle ABC)$ for $\text{area}(\blacktriangle ABC)$, since $\triangle ABC$ completely determines the solid triangle $\blacktriangle ABC$.

Proof. Draw the line m thru A which is parallel to (BC) and line n thru C parallel to (AB) . Note that $m \nparallel n$; set $D = m \cap n$. By construction, $\square ABCD$ is a parallelogram.



Note that $\blacksquare ABCD$ admits a subdivision into $\triangle ABC$ and $\triangle CDA$. Therefore,

$$\text{area}(\blacksquare ABCD) = \text{area}(\triangle ABC) + \text{area}(\triangle CDA)$$

Since $\square ABCD$ is a parallelogram, Lemma 7.16 implies that

$$AB = CD \quad \text{and} \quad BC = DA.$$

Therefore, by the SSS congruence condition, we have $\triangle ABC \cong \triangle CDA$. In particular

$$\text{area}(\triangle ABC) = \text{area}(\triangle CDA).$$

From above and Proposition 20.14, we get that

$$\begin{aligned} \text{area}(\triangle ABC) &= \frac{1}{2} \cdot \text{area}(\blacksquare ABCD) = \\ &= \frac{1}{2} \cdot h_A \cdot a \end{aligned}$$

$\square$

20.17. Exercise. Let h_A , h_B and h_C denote the altitudes of $\triangle ABC$ from vertices A , B and C correspondingly. Note that from Theorem 20.16, it follows that

$$h_A \cdot BC = h_B \cdot CA = h_C \cdot AB.$$

Give a proof of this statement without using area.

20.18. Exercise. Assume M lies inside the parallelogram $ABCD$; that is, M belongs to the solid parallelogram $\blacksquare ABCD$, but does not lie on its sides. Show that

$$\text{area}(\triangle ABM) + \text{area}(\triangle CDM) = \frac{1}{2} \cdot \text{area}(\blacksquare ABCD).$$

20.19. Exercise. Assume that diagonals of a nondegenerate quadrilateral $ABCD$ intersect at point M . Show that

$$\text{area}(\triangle ABM) \cdot \text{area}(\triangle CDM) = \text{area}(\triangle BCM) \cdot \text{area}(\triangle DAM).$$

20.20. Exercise. Let r be the inradius of $\triangle ABC$ and p be its semiperimeter; that is, $p = \frac{1}{2} \cdot (AB + BC + CA)$. Show that

$$\text{area}(\triangle ABC) = p \cdot r.$$

20.21. Advanced exercise. Show that for any affine transformation β there is a constant $k > 0$ such that the equality

$$\text{area}[\beta(\triangle)] = k \cdot \text{area } \triangle.$$

holds for any solid triangle $\triangle$.

Moreover, if β has the matrix form $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} v \\ w \end{pmatrix}$, then

$$k = |\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}| = |a \cdot d - b \cdot c|.$$

Area method

In this section we will give examples of slim proofs using the properties of area. Note that these proofs are not truly elementary since the price one pays to introduce the area function is high.

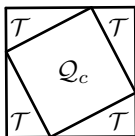
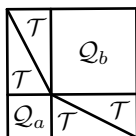
We start with the proof of the Pythagorean theorem. In the Elements of Euclid, the Pythagorean theorem was formulated as equality 8 below and the proof used a similar technique.

Proof. We need to show that if a and b are legs and c is the hypotenuse of a right triangle, then

$$a^2 + b^2 = c^2.$$

Let $\mathcal{T}$ denotes the right solid triangle with legs a and b and by $\mathcal{Q}_x$ be the solid square with side x .

Let us construct two subdivisions of $\mathcal{Q}_{a+b}$.



1. Subdivide $\mathcal{Q}_{a+b}$ into two solid squares congruent to $\mathcal{Q}_a$ and $\mathcal{Q}_b$ and 4 solid triangles congruent to $\mathcal{T}$, see the left diagram.

2. Subdivide $\mathcal{Q}_{a+b}$ into one solid square congruent to $\mathcal{Q}_c$ and 4 solid right triangles congruent to $\mathcal{T}$, see the right diagram.

Applying Proposition 20.11 few times, we get that

$$\begin{aligned} \text{area } \mathcal{Q}_{a+b} &= \text{area } \mathcal{Q}_a + \text{area } \mathcal{Q}_b + 4 \cdot \text{area } \mathcal{T} = \\ &= \text{area } \mathcal{Q}_c + 4 \cdot \text{area } \mathcal{T}. \end{aligned}$$

Therefore,

$$\text{area } \mathcal{Q}_a + \text{area } \mathcal{Q}_b = \text{area } \mathcal{Q}_c.$$

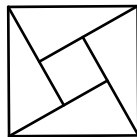
By Theorem 20.12,

$$\text{area } \mathcal{Q}_x = x^2,$$

for any $x > 0$. Hence the statement follows. $\square$

20.22. Exercise. Build another proof of the Pythagorean theorem based on the diagram.

(In the notations above it shows a subdivision of $\mathcal{Q}_c$ into $\mathcal{Q}_{a-b}$ and four copies of $\mathcal{T}$ if $a > b$.)



20.23. Exercise. Show that the sum of distances from a point to the sides of an equilateral triangle is the same for all points inside the triangle.

Let us prove Lemma 8.8 using the area method. That is, we need to show that if $\triangle ABC$ is nondegenerate and its angle bisector at A intersects $[BC]$ at the point D . Then

$$\frac{AB}{AC} = \frac{DB}{DC}.$$

In the proof we will use the following claim:

20.24. Claim. Assume that two triangles ABC and $A'B'C'$ in the Euclidean plane have equal altitudes dropped from A and A' correspondingly. Then

$$\frac{\text{area}(\triangle A'B'C')}{\text{area}(\triangle ABC)} = \frac{B'C'}{BC}.$$

In particular, the same identity holds if $A = A'$ and the bases $[BC]$ and $[B'C']$ lie on one line.

Proof. Let h be the altitude. By Theorem 20.16,

$$\frac{\text{area}(\triangle A'B'C')}{\text{area}(\triangle ABC)} = \frac{\frac{1}{2} \cdot h \cdot B'C'}{\frac{1}{2} \cdot h \cdot BC} = \frac{B'C'}{BC}. \quad \square$$

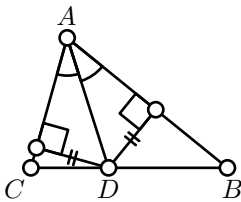
Proof of Lemma 8.8. Applying Claim 20.24, we get that

$$\frac{\text{area}(\triangle ABD)}{\text{area}(\triangle ACD)} = \frac{BD}{CD}.$$

By Proposition 8.10 the triangles ABD and ACD have equal altitudes from D . Applying Claim 20.24 again, we get that

$$\frac{\text{area}(\triangle ABD)}{\text{area}(\triangle ACD)} = \frac{AB}{AC}$$

and hence the result. $\square$



20.25. Exercise. Assume that the point X lies inside a nondegenerate triangle ABC . Show that X lies on the median from A if and only if

$$\text{area}(\triangle ABX) = \text{area}(\triangle ACX).$$

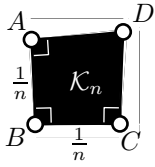
20.26. Exercise. Build a proof of Theorem 8.5 based on the Exercise 20.25.

Namely, show that medians of nondegenerate triangle intersect at one point and the point of their intersection divides each median in the ratio 1:2.

Area in the neutral planes and spheres

An analog of Theorem 20.7 holds in the neutral planes and spheres. In the formulation of this theorem, the solid unit square $\mathcal{K}$ has to be exchanged to a fixed nondegenerate polygonal set. One has to make such change for good reason — hyperbolic plane and sphere have no unit squares.

The set $\mathcal{K}$ in this case plays role of the unit measure for the area and changing $\mathcal{K}$ will require conversion of area units.



According to the standard convention, the set $\mathcal{K}$ is taken so that on small scales area behaves like area in the Euclidean plane. Say, if $\mathcal{K}_n$ denotes the solid quadrilateral $\blacksquare ABCD$ with right angles at A , B and C such that $AB = BC = \frac{1}{n}$, then we may assume that

$$\textcircled{9} \quad n^2 \cdot \text{area } \mathcal{K}_n \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

This convention works equally well for spheres and neutral planes, including Euclidean plane. In spherical geometry equivalently we may assume that if r is the radius of the sphere, then the area of whole sphere is $4 \cdot \pi \cdot r^2$.

Recall that *defect of triangle* $\triangle ABC$ is defined as

$$\text{defect}(\triangle ABC) := \pi - |\angle ABC| - |\angle BCA| - |\angle CAB|.$$

It turns out that any neutral plane or sphere there is a real number k such that

$$\textcircled{10} \quad k \cdot \text{area}(\blacktriangle ABC) + \text{defect}(\triangle ABC) = 0$$

for any $\triangle ABC$.

This number k is called *curvature*; $k = 0$ for the Euclidean plane, $k = -1$ for the h-plane and $k = 1$ for the unit sphere and $k = \frac{1}{r^2}$ for the sphere of radius r .

In particular, it follows that any ideal triangle in h-plane has area π . Similarly in the unit sphere the area of equilateral triangle with right angles has area $\frac{\pi}{2}$; since whole sphere can be subdivided in eight such triangles, we get that the area of unit sphere is $4 \cdot \pi$.

The identity $\textcircled{10}$ can be used as an alternative way to introduce area function; it works on spheres and all neutral planes, except for the Euclidean plane.

Quadrable sets

A set $\mathcal{S}$ in the plane is called *quadrable* if for any $\varepsilon > 0$ there are two polygonal sets $\mathcal{P}$ and $\mathcal{Q}$ such that

$$\mathcal{P} \subset \mathcal{S} \subset \mathcal{Q} \quad \text{and} \quad \text{area } \mathcal{Q} - \text{area } \mathcal{P} < \varepsilon.$$

If $\mathcal{S}$ is quadrable, its area can be defined as the necessarily unique real number $s = \text{area } \mathcal{S}$ such that the inequality

$$\text{area } \mathcal{Q} \leq s \leq \text{area } \mathcal{P}$$

holds for any polygonal sets $\mathcal{P}$ and $\mathcal{Q}$ such that $\mathcal{P} \subset \mathcal{S} \subset \mathcal{Q}$.

20.27. Exercise. *Let $\mathcal{D}$ be the unit disc; that is, $\mathcal{D}$ is a set which contains the unit circle Γ and all the points inside Γ .*

Show that $\mathcal{D}$ is a quadrable set.

Since $\mathcal{D}$ is quadrable, the expression $\text{area } \mathcal{D}$ makes sense and the constant π can be defined as $\pi = \text{area } \mathcal{D}$.

It turns out that the class of quadrable sets is the largest class for which the area can be defined in such a way that it satisfies all the conditions in Theorem 20.7 including uniqueness.

There is a way to define area for all bounded sets which satisfies all the conditions in the Theorem 20.7 excluding uniqueness. (A set in the plane is called *bounded* if it lies inside of a circle.)

In the hyperbolic plane and in the sphere there is no similar construction. If you wonder why, read about *doubling the ball paradox* of Felix Hausdorff, Stefan Banach and Alfred Tarski.

Hints

Exercise 1.2. Only the triangle inequality requires a proof — the rest of conditions in Definition 1.1 are evident. Let $A = (x_A, y_A)$, $B = (x_B, y_B)$ and $C = (x_C, y_C)$. Set

$$\begin{aligned}x_1 &= x_B - x_A, & y_1 &= y_B - y_A, \\x_2 &= x_C - x_B, & y_2 &= y_C - y_B.\end{aligned}$$

(a). The inequality

$$d_1(A, C) \leq d_1(A, B) + d_1(B, C)$$

can be written as

$$|x_1 + x_2| + |y_1 + y_2| \leq |x_1| + |y_1| + |x_2| + |y_2|.$$

The latter follows since $|x_1 + x_2| \leq |x_1| + |x_2|$ and $|y_1 + y_2| \leq |y_1| + |y_2|$.

(b). The inequality

❶
$$d_2(A, C) \leq d_2(A, B) + d_2(B, C)$$

can be written as

$$\sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2} \leq \sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2 + y_2^2}.$$

Take the square of the left and the right hand sides, simplify, take the square again and simplify again. You should get the following inequality:

$$0 \leq (x_1 \cdot y_2 - x_2 \cdot y_1)^2,$$

which is equivalent to ❶ and evidently true.

(c). The inequality

$$d_\infty(A, C) \leq d_\infty(A, B) + d_\infty(B, C)$$

can be written as

❷
$$\max\{|x_1 + x_2|, |y_1 + y_2|\} \leq \max\{|x_1|, |y_1|\} + \max\{|x_2|, |y_2|\}.$$

Without loss of generality, we may assume that

$$\max\{|x_1 + x_2|, |y_1 + y_2|\} = |x_1 + x_2|.$$

Further,

$$|x_1 + x_2| \leq |x_1| + |x_2| \leq \max\{|x_1|, |y_1|\} + \max\{|x_2|, |y_2|\}.$$

Hence ❷ follows.

Exercise 1.4. If $A \neq B$, then $d_X(A, B) > 0$. Since f is distance-preserving,

$$d_Y(f(A), f(B)) = d_X(A, B).$$

Therefore, $d_Y(f(A), f(B)) > 0$; hence $f(A) \neq f(B)$.

Exercise 1.5. Set $f(0) = a$ and $f(1) = b$. Note that $b = a + 1$ or $a - 1$. Moreover, $f(x) = a \pm x$ and at the same time, $f(x) = b \pm (x - 1)$ for any x .

If $b = a + 1$, it follows that $f(x) = a + x$ for any x .

The same way, if $b = a - 1$, it follows that $f(x) = a - x$ for any x .

Exercise 1.6. Show that the map $(x, y) \mapsto (x + y, x - y)$ is an isometry $(\mathbb{R}^2, d_1) \rightarrow (\mathbb{R}^2, d_\infty)$. That is, you need to check if this map is bijective and distance-preserving.

Exercise 1.7. First prove that *two points $A = (x_A, y_A)$ and $B = (x_B, y_B)$ on the Manhattan plane have a unique midpoint if and only if $x_A = x_B$ or $y_A = y_B$* ; compare with the example on page 17.

Then use above statement to prove that any motion of the Manhattan plane can be written in one of the following two ways:

$$(x, y) \mapsto (\pm x + a, \pm y + b) \quad \text{or} \quad (x, y) \mapsto (\pm y + b, \pm x + a),$$

for some fixed real numbers a and b . (In each case we have 4 choices of signs, so for fixed pair (a, b) we have 8 distinct motions.)

Exercise 1.9. Assume three points A , B and C lie on one line. Note that in this case one of the triangle inequalities with the points A , B and C becomes an equality.

Set $A = (-1, 1)$, $B = (0, 0)$ and $C = (1, 1)$. Show that for d_1 and d_2 all the triangle inequalities with the points A , B and C are strict. It follows that the graph is not a line.

For d_∞ show that $(x, |x|) \mapsto x$ gives the isometry of the graph to $\mathbb{R}$. Conclude that the graph is a line in $(\mathbb{R}^2, d_\infty)$.

Exercise 1.10. Applying the definition of lines, the problems are reduced to the following:

Assume that $a \neq b$, find the number of solutions for each of the following two equations:

$$|x - a| = |x - b| \quad \text{and} \quad |x - a| = 2|x - b|.$$

Each can be solved by taking the square of the left and the right hand sides. The numbers of solutions are 1 and 2 correspondingly.

Exercise 1.11. Fix an isometry $f: (PQ) \rightarrow \mathbb{R}$ such that $f(P) = 0$ and $f(Q) = q > 0$.

Assume that $f(X) = x$. By the definition of the half-line $X \in [PQ)$ if and only if $x \geq 0$. Show that the latter holds if and only if

$$|x - q| = ||x| - |q||.$$

Hence the result will follow.

Exercise 1.12. The equation $2\cdot\alpha \equiv 0$ means that $2\cdot\alpha = 2\cdot k\cdot\pi$ for some integer k . Therefore, $\alpha = k\cdot\pi$ for some integer k .

Equivalently, $\alpha = 2\cdot n\cdot\pi$ or $\alpha = (2\cdot n + 1)\cdot\pi$ for some integer n . The first identity means that $\alpha \equiv 0$ and the second means that $\alpha \equiv \pi$.

Exercise 1.13. (a). By the triangle inequality,

$$|f(A') - f(A)| \leq d(A', A).$$

Therefore, we can take $\delta = \varepsilon$.

(b). By the triangle inequality,

$$\begin{aligned} |f(A', B') - f(A, B)| &\leq |f(A', B') - f(A, B')| + |f(A, B') - f(A, B)| \\ &\leq d(A', A) + d(B', B). \end{aligned}$$

Therefore, we can take $\delta = \frac{\varepsilon}{2}$.

Exercise 1.14. Fix $A \in \mathcal{X}$ and $B \in \mathcal{Y}$ such that $f(A) = B$.

Fix $\varepsilon > 0$. Since g is continuous at B , there is a positive value δ_1 such that

$$d_{\mathcal{Z}}(g(B'), g(B)) < \varepsilon \quad \text{if} \quad d_{\mathcal{Y}}(B', B) < \delta_1.$$

Since f is continuous at A , there is $\delta_2 > 0$ such that

$$d_{\mathcal{Y}}(f(A'), f(A)) < \delta_1 \quad \text{if} \quad d_{\mathcal{X}}(A', A) < \delta_2.$$

Since $f(A) = B$, we get that

$$d_{\mathcal{Z}}(h(A'), h(A)) < \varepsilon \quad \text{if} \quad d_{\mathcal{X}}(A', A) < \delta_2.$$

Hence the result.

Exercise 2.1. By Axiom I, there are at least two points in the plane. Therefore, by Axiom II, the plane contains a line. To prove (a), it remains to note that line is an infinite set of points. To prove (b) apply in addition Axiom III.

Exercise 2.3. By Axiom II, $(OA) = (OA')$. Therefore, the statement boils down to the following:

Assume $f: \mathbb{R} \rightarrow \mathbb{R}$ is a motion of the line which sends $0 \rightarrow 0$ and one positive number to a positive number, then f is an identity map.

The latter follows from Exercise 1.5.

Exercise 2.6. By Proposition 2.5, $\angle AOA = 0$. It remains to apply Axiom IIIa.

Exercise 2.10. Apply Proposition 2.5, Theorem 2.8 and Exercise 1.12.

Exercise 2.11. By Axiom IIIb,

$$2\cdot\angle BOC \equiv 2\cdot\angle AOC - 2\cdot\angle AOB \equiv 0.$$

By Exercise 1.12, it implies that $\angle BOC$ is either 0 or π . It remains to apply Exercise 2.6 and Theorem 2.8 correspondingly in these two cases.

Exercise 2.12. Fix two points A and B provided by Axiom I.

Fix a real number $0 < \alpha < \pi$. By Axiom IIIa there is a point C such that $\angle ABC = \alpha$. Use Proposition 2.2 to show that $\triangle ABC$ is nondegenerate.

Exercise 2.14. Applying Proposition 2.13, we get that $\angle AOC = \angle BOD$. It remains to apply Axiom IV.

Exercise 3.1. Set $\alpha = \angle AOB$ and $\beta = \angle BOA$. Note that $\alpha = \pi$ if and only if $\beta = \pi$. Otherwise $\alpha = -\beta$. Hence the result.

Exercise 3.3. Set $\alpha = \angle AOB$, $\beta = \angle BOC$ and $\gamma = \angle COA$. By Axiom IIIb and Proposition 2.5, we have

$$\textcircled{1} \quad \alpha + \beta + \gamma \equiv 0.$$

Note that $0 < \alpha + \beta < 2 \cdot \pi$ and $|\gamma| \leq \pi$. If $\gamma > 0$, then $\textcircled{1}$ implies

$$\alpha + \beta + \gamma = 2 \cdot \pi$$

and if $\gamma < 0$, then $\textcircled{1}$ implies

$$\alpha + \beta + \gamma = 0.$$

Exercise 3.11. Note that O and A' lie on the same side of (AB) . Analogously O and B' lie on the same side of (AB) . Hence the result.

Exercise 3.13. Apply Theorem 3.7 for $\triangle PQX$ and $\triangle PQY$ and then apply Corollary 3.10a.

Exercise 3.14. Note that it is sufficient to consider the cases when $A' \neq B, C$ and $B' \neq A, C$.

Apply Pasch's theorem (3.12) twice: (1) for $\triangle AA'C$ and (BB') , and (2) for $\triangle BB'C$ and (AA') .

Exercise 3.15. Assume that Z is the point of intersection.

Note that $Z \neq P$ and $Z \neq Q$. Therefore, $Z \notin (PQ)$.

Show that Z and X lie on one side of (PQ) . Repeat the argument to show that Z and Y lie on one side of (PQ) . It follows that X and Y lie on the same side of (PQ) — a contradiction.

Exercise 3.20. The “only-if” part follows from the triangle inequality. To prove “if” part apply Theorem 3.17 for the values r_1 , r_2 and d .

Exercise 4.3. Apply Theorem 4.2 twice.

Exercise 4.5. Consider the points D and D' , such that M is the midpoint of $[AD]$ and M' is the midpoint of $[A'D']$. Show that $\triangle ABD \cong \triangle A'B'D'$ and use it to prove that $\triangle A'B'C' \cong \triangle ABC$.

Exercise 4.6. (a) Apply SAS.

(b) Use (a) and apply SSS.

Exercise 4.7. Choose $B' \in [AC]$ such that $AB = AB'$. Note that $BC = B'C$. By SSS, $\triangle ABC \cong \triangle AB'C$.

Exercise 4.8. Without loss of generality, we may assume that X is distinct from A , B and C . Set $f(X) = X'$; assume $X' \neq X$.

Note that $AX = AX'$, $BX = BX'$ and $CX = CX'$. By SSS we get that $\angle ABX = \pm \angle ABX'$. Since $X \neq X'$, we get that

$$\angle ABX \equiv -\angle ABX'.$$

The same way we get that

$$\angle CBX \equiv -\angle CBX'.$$

Subtracting these two identities from each other, we get that

$$\angle ABC \equiv -\angle ABC.$$

Conclude that $\angle ABC = 0$ or π . That is, $\triangle ABC$ is degenerate — a contradiction.

Exercise 5.1. By Axiom IIIb and Theorem 2.8, we have

$$\angle XO A - \angle XO B \equiv \pi.$$

Since $|\angle XO A|, |\angle XO B| \leq \pi$, we get that

$$|\angle XO A| + |\angle XO B| = \pi.$$

Hence the statement follows.

Exercise 5.3. Assume X and A lie on the same side of ℓ .

Note that A and B lie on opposite sides of ℓ . Therefore, by Corollary 3.10, $[AX]$ does not intersect ℓ and $[BX]$ intersects ℓ ; let Y denotes the intersection point.

Note that $Y \notin [AX]$. By Exercise 4.7,

$$BX = AY + YX > AX.$$

This way we proved the “if” part. To prove the “only if” part, it remains to switch A and B , repeat the above argument and apply Theorem 5.2.

Exercise 5.4. Apply Exercise 5.3, Theorem 4.1 and Exercise 3.3.

Exercise 5.8. Choose an arbitrary nondegenerate triangle ABC . Let $\triangle \hat{A}\hat{B}\hat{C}$ denotes its image after the motion.

If $A \neq \hat{A}$, apply the reflection in the perpendicular bisector of $[A\hat{A}]$. This reflection sends A to $\hat{A}$. Let B' and C' denote the reflections of B and C correspondingly.

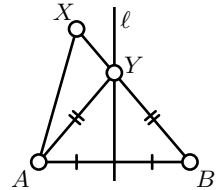
If $B' \neq \hat{B}$, apply the reflection in the perpendicular bisector of $[B'\hat{B}]$. This reflection sends B' to $\hat{B}$. Note that $\hat{A}\hat{B} = \hat{A}B'$; that is, $\hat{A}$ lies on the perpendicular bisector. Therefore, $\hat{A}$ reflects to itself. Let C'' denotes the reflection of C' .

Finally, if $C'' \neq \hat{C}$, apply the reflection in $(\hat{A}\hat{B})$. Note that $\hat{A}\hat{C} = \hat{A}C''$ and $\hat{B}\hat{C} = \hat{B}C''$; that is, $(\hat{A}\hat{B})$ is the perpendicular bisector of $[C''\hat{C}]$. Therefore, this reflection sends C'' to $\hat{C}$.

Apply Exercise 4.8 to show that the composition of the constructed reflections coincides with the given motion.

Exercise 5.9. Note that $\angle XBA = \angle ABP$, $\angle PBC = \angle CBY$. Therefore,

$$\angle XBY \equiv \angle XBP + \angle PBY \equiv 2 \cdot (\angle ABP + \angle PBC) \equiv 2 \cdot \angle ABC.$$

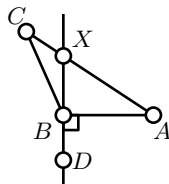


Exercise 5.11. If $\angle ABC$ is right, the statement follows from Lemma 5.10. Therefore, we can assume that $\angle ABC$ is obtuse.

Draw a line (BD) perpendicular to (BA) . Since $\angle ABC$ is obtuse, the angles DBA and DBC have opposite signs.

By Corollary 3.10, A and C lie on opposite sides of (BD) . In particular, $[AC]$ intersects (BD) at a point; denote it by X .

Note that $AX < AC$ and by Lemma 5.10, $AB \leq AX$.



Exercise 5.12. Let O be the center of the circle. Note that we can assume that $O \neq P$.

Assume P lies between X and Y . By Exercise 5.1, we can assume that $\angle OPX$ is right or obtuse. By Exercise 5.11, $OP < OX$; that is, P lies inside Γ .

If P does not lie between X and Y , we can assume that X lies between P and Y . Since $OX = OY$, Exercise 5.11 implies that $\angle OXY$ is acute. Therefore, $\angle OXP$ is obtuse. Applying Exercise 5.11 again we get that $OP > OX$; that is, P lies outside Γ .

Exercise 5.13. Apply Theorem 5.2.

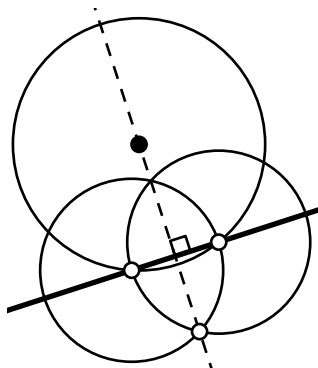
Exercise 5.15. Use Exercise 5.13 and Theorem 5.5.

Exercise 5.17. Let P' be the reflection of P in (OO') . Note that P' lies on both circles and $P' \neq P$ if and only if $P \notin (OO')$.

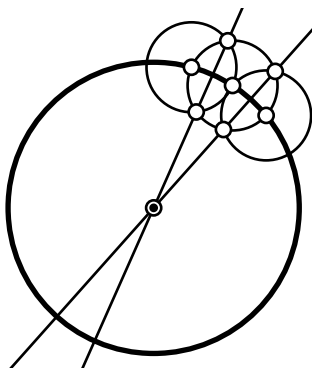
Exercise 5.18. Apply Exercise 5.17.

Exercise 5.19. Let A and B be the points of intersection. Note that the centers lie on the perpendicular bisector of the segment $[AB]$.

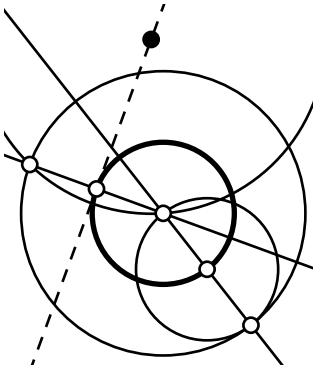
Exercise 5.21.



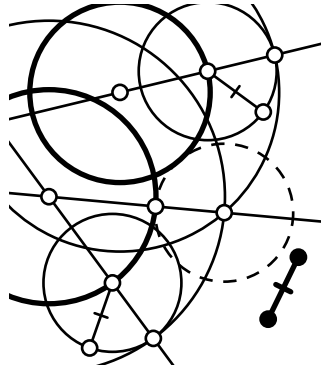
Exercise 5.22.



Exercise 5.23.



Exercise 5.24.



Exercise 6.3. By the AA similarity condition, the transformation multiplies the sides of any nondegenerate triangle by some number which may depend on the triangle.

Note that for any two nondegenerate triangles which share one side this number is the same. Applying this observation to a chain of triangles leads to a solution.

Exercise 6.5. Apply that $\triangle ADC \sim \triangle CDB$.

Exercise 6.6. Apply the Pythagorean theorem (6.4) and the SSS congruence condition.

Exercise 6.7. By the AA similarity condition (6.2), $\triangle AYC \sim \triangle BXC$.

Conclude that

$$\frac{YC}{AC} = \frac{XC}{BC}.$$

Apply the SAS similarity condition to show that $\triangle ABC \sim \triangle YXC$.

Use AA and equality of vertical angles to prove that $\triangle AZX \sim \triangle BZY$.

Exercise 7.4. Apply Proposition 7.1 to show that $k \parallel m$. By Corollary 7.3, $k \parallel n \Rightarrow m \parallel n$. The latter contradicts that $m \perp n$.

Exercise 7.5. Repeat the construction in Exercise 5.21 twice.

Exercise 7.8. By the transversal property 7.6,

$$\angle B'BC \equiv \pi - \angle C'B'B.$$

Since B' lies between A and B , we get that $\angle ABC = \angle B'BC$ and $\angle AB'C' + \angle C'B'B \equiv \pi$. Hence $\angle ABC = \angle AB'C'$.

The same way we can prove that $\angle BCA = \angle B'C'A$. It remains to apply the AA similarity condition.

Exercise 7.9. Assume we need to trisect segment $[AB]$. Construct a line $\ell \neq (AB)$ with four points A, C_1, C_2, C_3 such that C_1 and C_2 trisect $[AC_3]$. Draw the line (BC_3) and draw parallel lines thru C_1 and C_2 . The points of intersections of these two lines with (AB) trisect the segment $[AB]$.

Exercise 7.11. Apply twice Theorem 4.2 and twice Theorem 7.10.

Exercise 7.12. If $\triangle ABC$ is degenerate, then one of the angle measures is π and the other two are 0. Hence the result.

Assume $\triangle ABC$ is nondegenerate. Set $\alpha = \angle CAB$, $\beta = \angle ABC$ and $\gamma = \angle BCA$.

By Theorem 3.7, we may assume that $0 < \alpha, \beta, \gamma < \pi$. Therefore,

$$\textcircled{1} \quad 0 < \alpha + \beta + \gamma < 3 \cdot \pi.$$

By Theorem 7.10,

$$\textcircled{2} \quad \alpha + \beta + \gamma \equiv \pi.$$

From $\textcircled{1}$ and $\textcircled{2}$ the result follows.

Exercise 7.13. Apply twice Theorem 4.2 and once Theorem 7.10.

Exercise 7.14. Let O denotes the center of the circle.

Note that $\triangle AOX$ is isosceles and $\angle OXC$ is right. Applying 7.10 and 4.2 and simplifying, you should get

$$4 \cdot \angle CAX \equiv \pi.$$

Show that $\angle CAX$ has to be acute. It follows then that $\angle CAX = \pm \frac{\pi}{4}$.

Exercise 7.15. Apply Theorem 7.10 to $\triangle ABC$ and $\triangle BDA$.

Exercise 7.17. Since $\triangle ABC$ is isosceles, $\angle CAB = \angle BCA$.

By SSS, $\triangle ABC \cong \triangle CDA$. Therefore,

$$\pm \angle DCA = \angle BCA = \angle CAB.$$

Since $D \neq C$, we get “ $-$ ” in the last formula. Use the transversal property (7.6) to show that $(AB) \parallel (CD)$. Repeat the argument to show that $(AD) \parallel (BC)$

Exercise 7.18. Apply Lemma 7.16 together with the transversal property (7.6) to the diagonals and a pair of opposite sides. After that use the ASA-congruence condition (4.1).

Exercise 7.19. By Lemma 7.16 and SSS,

$$AC = BD \quad \Longleftrightarrow \quad \angle ABC = \pm \angle BCD.$$

By the transversal property (7.6),

$$\angle ABC + \angle BCD \equiv \pi.$$

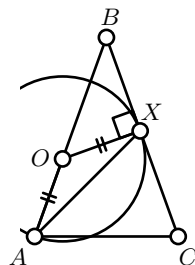
Therefore,

$$AC = BD \quad \Longleftrightarrow \quad \angle ABC = \angle BCD = \pm \frac{\pi}{2}.$$

Exercise 7.20. Fix a parallelogram $ABCD$. By Exercise 7.18, its diagonals $[AC]$ and $[BD]$ have a common midpoint; denote it by M .

Use SSS and Lemma 7.16 to show that

$$AB = CD \quad \Longleftrightarrow \quad \triangle AMB \cong \triangle AMD \quad \Longleftrightarrow \quad \angle AMB = \pm \frac{\pi}{2}.$$



Exercise 7.21. (a). Use the uniqueness of the parallel line (Theorem 7.2).

(b) Use Lemma 7.16 and the Pythagorean theorem (6.4).

Exercise 7.22. Set $A = (0, 0)$, $B = (c, 0)$ and $C = (x, y)$. Clearly, $AB = c$, $AC^2 = x^2 + y^2$ and $BC^2 = (c - x)^2 + y^2$.

It remains to show that there is a pair of real numbers (x, y) which satisfy the following system of equations:

$$\begin{cases} b^2 = x^2 + y^2 \\ a^2 = (c - x)^2 + y^2 \end{cases}$$

if $0 < a \leq b \leq c \leq a + c$.

Exercise 7.23. Let $M = (x, y)$. Note that $MA = MB$ if and only if

$$(x - x_A)^2 + (y - y_A)^2 = (x - x_B)^2 + (y - y_B)^2.$$

To prove the first, simplify this equation. For the second part use that any line is a perpendicular bisector to some line segment.

Exercise 7.24. Rewrite the equation as

$$(x + \frac{a}{2})^2 + (y + \frac{b}{2})^2 = (\frac{a}{2})^2 + (\frac{b}{2})^2 - c$$

and think.

Exercise 7.25. We can choose the coordinates so that $B = (0, 0)$ and $A = (a, 0)$ for some $a > 0$. If $M = (x, y)$, then the equation $AM = k \cdot BM$ can be written in coordinates as

$$k^2 \cdot (x^2 + y^2) = (x - a)^2 + y^2.$$

It remains to rewrite this equation as in Exercise 7.24.

Exercise 8.2. Apply Theorem 8.1 and Theorem 5.2.

Exercise 8.4. Note that $(AC) \perp (BH)$ and $(BC) \perp (AH)$ and apply Theorem 8.3.

Exercise 8.6. Use the idea from the proof of Theorem 8.5 to show that $(XY) \parallel (AC) \parallel (VW)$ and $(XV) \parallel (BD) \parallel (YW)$.

Exercise 8.7. Let (BX) and (BY) be the internal and external bisectors of $\angle ABC$. Then

$$\begin{aligned} 2 \cdot \angle XBY &\equiv 2 \cdot \angle XBA + 2 \cdot \angle ABY \equiv \\ &\equiv \angle CBA + \pi + 2 \cdot \angle ABC \equiv \\ &\equiv \pi + \angle CBC = \pi \end{aligned}$$

and hence the result.

Exercise 8.9. If E is the point of intersection of (BC) with the external bisector of $\angle BAC$, then

$$\frac{AB}{AC} = \frac{EB}{EC}.$$

It can be proved along the same lines as Lemma 8.8.

Exercise 8.12. Apply Lemma 8.8. Also see the solution of Exercise 11.2.

Exercise 8.13. Apply ASA for the two triangles which the bisector cuts from the original triangle.

Exercise 8.14. Let I be the incenter. By SAS, we get that $\triangle AIZ \cong \triangle AIY$. Therefore, $AY = AZ$. The same way we get that $BX = BZ$ and $CX = CY$. Hence the result.

Exercise 8.15. Let $\triangle ABC$ be the given acute triangle and $\triangle A'B'C'$ be its orthic triangle. Note that $\triangle AA'C \sim \triangle BB'C$. Use it to show that $\triangle A'B'C' \sim \triangle ABC$.

The same way we get that $\triangle AB'C' \sim \triangle ABC$. It follows that $\angle A'B'C = \angle AB'C'$. Conclude that (BB') bisects $\angle A'B'C'$.

If $\triangle ABC$ is obtuse, then its orthocenter coincides with one of the *excenters* of $\triangle ABC$; that is, the point of intersection of two external and one internal bisectors of $\triangle ABC$.

Exercise 8.16. Apply 4.2, 7.6 and 7.16.

Exercise 9.3. (a). Apply Theorem 9.2 for $\angle XX'Y$ and $\angle X'YY'$ and Theorem 7.10 for $\triangle PYX'$.

(b) If P is inside of Γ then P lies between X and X' and between Y and Y' in this case $\angle XPY$ is vertical to $\angle X'PY'$. If P is outside of Γ then $[PX] = [PX']$ and $[PY] = [PY']$. In both cases we have that $\angle XPY = \angle X'PY'$.

Applying Theorem 9.2 and Exercise 2.11, we get that

$$2 \cdot \angle Y'X'P \equiv 2 \cdot \angle Y'X'X \equiv 2 \cdot \angle Y'YX \equiv 2 \cdot \angle PYX.$$

According to Theorem 3.7, $\angle Y'X'P$ and $\angle PYX$ have the same sign; therefore

$$\angle Y'X'P = \angle PYX.$$

It remains to apply the AA similarity condition.

(c) Apply (b) assuming $[YY']$ is the diameter of Γ .

Exercise 9.4. Apply Exercise 9.3b three times.

Exercise 9.5. Let X any Y be the foot points of the altitudes from A and B . Let O denotes the circumcenter.

By AA condition, $\triangle AXC \sim \triangle BYC$. Thus

$$\angle A'OC \equiv 2 \cdot \angle A'AC \equiv -2 \cdot \angle B'BC \equiv -\angle B'OC.$$

By SAS, $\triangle A'OC \cong \triangle B'OC$. Therefore, $A'C = B'C$.

Exercise 9.7. Construct the circles Γ and Γ' on the diameters $[AB]$ and $[A'B']$ correspondingly. By Corollary 9.6, any point Z in the intersection $\Gamma \cap \Gamma'$ will do.

Exercise 9.8. Note that $\angle AA'B = \pm \frac{\pi}{2}$ and $\angle AB'B = \pm \frac{\pi}{2}$. Then apply Theorem 9.10 to $\square AA'BB'$.

If O is the center of the circle, then

$$\angle AOB \equiv 2 \cdot \angle AA'B \equiv \pi.$$

That is, O is the midpoint of $[AB]$.

Exercise 9.9. Guess the construction from the diagram. To prove it, apply Theorem 8.3 and Corollary 9.6.

Exercise 9.11. Apply the transversal property (7.6) and the theorem on inscribed angles (9.2).

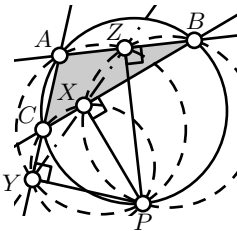
Exercise 9.12. Apply Theorem 9.10 twice for $\square ABYX$ and $\square ABY'X'$ and use the transversal property (7.6).

Exercise 9.14. One needs to show that the lines $(A'B')$ and (XP) are not parallel, otherwise the first line in the proof does not make sense.

In addition, the following identities:

$$2 \cdot \angle AXP \equiv 2 \cdot \angle AX Y, \quad 2 \cdot \angle ABP \equiv 2 \cdot \angle ABB', \quad 2 \cdot \angle AA'B' \equiv 2 \cdot \angle AA'Y.$$

Exercise 9.15. By Corollary 9.6, the points L , M and N lie on the circle Γ with diameter $[OX]$. It remains to apply Theorem 9.2 for the circle Γ and two inscribed angles with vertex at O .



Advanced exercise 9.16. Let X , Y and Z denote the foot points of P on (BC) , (CA) and (AB) correspondingly.

Notice that $\square AZPY$, $\square BXPZ$, $\square CYPX$ and $\square ABPC$ are inscribed. Therefore

$$\begin{aligned} 2 \cdot \angle CXY &\equiv 2 \cdot \angle CPY, & 2 \cdot \angle BXZ &\equiv 2 \cdot \angle BPZ, \\ 2 \cdot \angle YAZ &\equiv 2 \cdot \angle YPZ, & 2 \cdot \angle CAB &\equiv 2 \cdot \angle CPB. \end{aligned}$$

Conclude that $2 \cdot \angle CXY \equiv 2 \cdot \angle BXZ$ and hence X , Y and Z lie on one line.

Exercise 9.20. By Theorem 7.10,

$$\angle ABC + \angle BCA + \angle CAB \equiv \pi.$$

It remains to apply Proposition 9.19 twice.

Exercise 9.21. If $C \in (AX)$, then the arc is the line segment $[AC]$ or the union of two half-lines in (AX) with vertices at A and C .

Assume $C \notin (AX)$. Let ℓ be the perpendicular line dropped from A to (AX) and m be the perpendicular bisector of $[AC]$.

Note that $\ell \nparallel m$; set $O = \ell \cap m$. Note that the circle with center O passing thru A is also passing thru C and tangent to (AX) .

Note that one of the two arcs with endpoints A and C is tangent to (AX) .

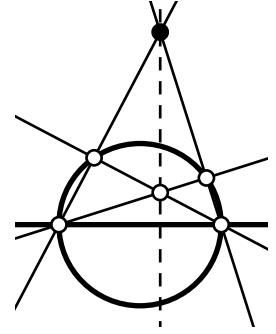
The uniqueness follows from the propositions 9.18 and 9.19.

Exercise 9.22. Use 9.19 and 7.10 to show that

$$\angle XAY = \angle ACY.$$

By Axiom IIIc, $\angle ACY \rightarrow 0$ as $AY \rightarrow 0$; hence the result.

Exercise 9.23. Apply Proposition 9.19 twice.



Exercise 9.24. Guess the construction from the diagram. To show that it produces the needed point, apply Theorem 9.2.

Exercise 10.1. By Lemma 5.16, $\angle OTP'$ is right. Therefore, $\triangle OPT \sim \triangle OTP'$ and in particular

$$OP \cdot OP' = OT^2$$

and hence the result.

Exercise 10.3. Let O denotes the center of Γ . Suppose $X, Y \in \Gamma$; in particular, $OX = OY$.

Note that the inversion sends X and Y to themselves. By Lemma 10.2,

$$\triangle OPX \sim \triangle OXP' \quad \text{and} \quad \triangle OPY \sim \triangle OYP'.$$

Therefore,

$$\frac{PX}{P'X} = \frac{OP}{OX} = \frac{OP}{OY} = \frac{PY}{P'Y}$$

and hence the result.

Exercise 10.4. By Lemma 10.2,

$$\begin{aligned} \angle IA'B' &\equiv -\angle IBA, & \angle IB'C' &\equiv -\angle ICB, & \angle IC'A' &\equiv -\angle IAC, \\ \angle IB'A' &\equiv -\angle IAB, & \angle IC'B' &\equiv -\angle IBC, & \angle IA'C' &\equiv -\angle ICA. \end{aligned}$$

It remains to apply the theorem on the sum of angles of triangle (7.10) to show that $(A'I) \perp (B'C')$, $(B'I) \perp (C'A')$ and $(C'I) \perp (B'A')$.

Exercise 10.5. Guess the construction from the diagram (the two nonintersecting lines on the diagram are parallel).

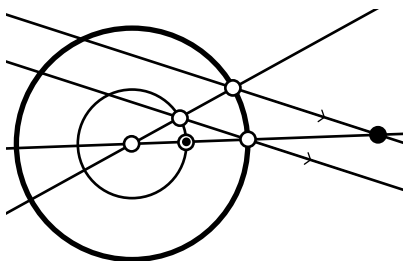
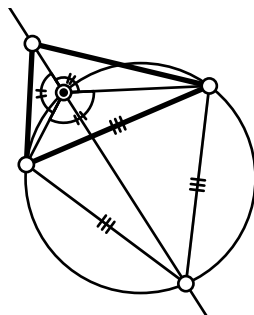
Exercise 10.8. First show that for any $r > 0$ and for any real numbers x, y distinct from 0,

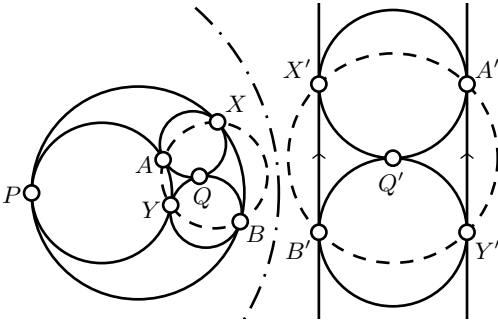
$$\frac{r^2}{(x+y)/2} = \left(\frac{r^2}{x} + \frac{r^2}{y} \right) / 2$$

if and only if $x = y$.

Let ℓ denotes the line passing thru Q, Q' and the center of the inversion O . Choose an isometry $\ell \rightarrow \mathbb{R}$ which sends O to 0; assume $x, y \in \mathbb{R}$ are the values of ℓ for the two points of intersection $\ell \cap \Gamma$; note that $x \neq y$. Assume r is the radius of the circle of inversion. Then the left hand side above is the coordinate of Q' and the right hand side is the coordinate of the center of Γ' .

Exercise 10.9. A solution is given on page 157.





Exercise 10.10. Apply an inversion in a circle with the center at the only point of intersection of the circles; then use Theorem 10.11.

Exercise 10.13. Label the points of tangency by X, Y, A, B, P and Q as on the diagram above. Apply an inversion with the center at P . Observe that the two circles which tangent at P become parallel lines and the remaining two circles are tangent to each other and these two

parallel lines.

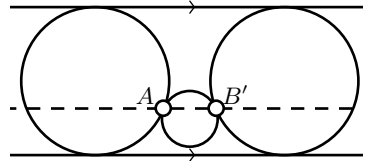
Note that the points of tangency A', B', X' and Y' with the parallel lines are vertexes of a square; in particular they lie on one circle. These points are images of A, B, X and Y under the inversion. By Theorem 10.7, the points A, B, X and Y also lie on one circline.

Advanced exercise 10.14. Apply the inversion in a circle with center A . The point A will go to infinity, the two circles tangent at A will become parallel lines and the two parallel lines will become circles tangent at A ; see the diagram.

It remains to show that the dashed line AB' is parallel to the other two lines.

Exercise 10.19. Let P_1 and P_2 be the inverses of P in Ω_1 and Ω_2 . Note that the points P, P_1 and P_2 are mutually distinct.

Use Theorem 8.1, to show that there is unique circline Γ which passes thru P, P_1 and P_2 . Use Corollary 10.17 to show that $\Gamma \perp \Omega_1$ and $\Gamma \perp \Omega_2$. Use Theorem 10.15 to prove uniqueness.



Exercise 10.20. Apply Theorem 10.6*b*, Exercise 7.15 and Theorem 9.2.

Exercise 10.21. Let T denotes a point of intersection of Ω_1 and Ω_2 . Let P be the foot point of T on (O_1O_2) . Show that

$$\triangle O_1PT \sim \triangle O_1TO_2 \sim \triangle TPO_2.$$

Conclude that P coincides with the inverses of O_1 in Ω_2 and of O_2 in Ω_1 .

Exercise 10.22. Since $\Gamma \perp \Omega_1$ and $\Gamma \perp \Omega_2$, Corollary 10.16 implies that the circles Ω_1 and Ω_2 are inverted in Γ to themselves.

Therefore, the points A and B are inverses of each other.

Since $\Omega_3 \ni A, B$, Corollary 10.17 implies that $\Omega_3 \perp \Gamma$.

Exercise 10.23. Follow the solution of Exercise 10.19.

Exercise 11.2. Let D denotes the midpoint of $[BC]$. Assume (AD) is the angle bisector at A .

Let $A' \in [AD)$ be the point distinct from A such that $AD = A'D$. Note that $\triangle CAD \cong \triangle BA'D$. In particular, $\angle BAA' = \angle AA'B$. It remains to apply Theorem 4.2 for $\triangle ABA'$.

Exercise 11.3. The statement is evident if A, B, C and D lie on one line.

In the remaining case, let O denotes the circumcenter. Apply theorem about isosceles triangle (4.2) to the triangles AOB , BOC , COD , DOA .

(Note that in the Euclidean plane the statement follows from Theorem 9.10 and Exercise 7.15, but one cannot use these statements in the neutral plane.)

Exercise 11.5. Arguing by contradiction, assume

$$2 \cdot (\angle ABC + \angle BCD) \equiv 0,$$

but $(AB) \nparallel (CD)$. Let Z be the point of intersection of (AB) and (CD) .

Note that

$$2 \cdot \angle ABC \equiv 2 \cdot \angle ZBC,$$

$$2 \cdot \angle BCD \equiv 2 \cdot \angle BCZ.$$

Apply Proposition 11.4 to $\triangle ZBC$ and try to arrive to a contradiction.

Exercise 11.6. Let $C'' \in [B'C']$ be the point such that $B'C'' = BC$.

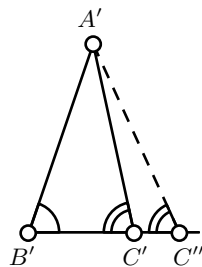
Note that by SAS, $\triangle ABC \cong \triangle A'B'C''$. Conclude that $\angle B'C'A' = \angle B'C''A'$.

Therefore, it is sufficient to show that $C'' = C'$. If $C' \neq C''$ apply Proposition 11.4 to $\triangle A'C'C''$ and try to arrive to a contradiction.

Exercise 11.7. Use Exercise 5.4 and Proposition 11.4.

Alternatively, use the same argument as in the solution of Exercise 5.12.

Exercise 11.10. Note that $|\angle ADC| + |\angle CDB| = \pi$. Then apply the definition of the defect.



Exercise 12.1. Let A and B be the ideal points of the h-line ℓ . Note that the center of the Euclidean circle containing ℓ lies at the intersection of the lines tangent to the absolute at the ideal points of ℓ .

Exercise 12.2. Assume A is an ideal point of the h-line ℓ and $P \in \ell$. Let P' denotes the inverse of P in the absolute. By Corollary 10.16, ℓ lies in the intersection of the h-plane and the (necessarily unique) circline passing thru P , A and P' .

Exercise 12.3. Let Ω and O denote the absolute and its center.

Let Γ be the circline containing $[PQ]_h$. Note that $[PQ]_h = [PQ]$ if and only if Γ is a line.

Let P' denotes the inverse of P in Ω . Note that O , P and P' lie on one line.

By the definition of h-line, $\Omega \perp \Gamma$. By Corollary 10.16, Γ passes thru P and P' . Therefore, Γ is a line if and only if it pass thru O .

Exercise 12.4. Assume that the absolute is a unit circle.

Set $a = OX = OY$. Note that $0 < a < \frac{1}{2}$ and

$$OX_h = \ln \frac{1+a}{1-a},$$

$$XY_h = \ln \frac{(1+2 \cdot a) \cdot (1-a)}{(1-2 \cdot a) \cdot (1+a)}.$$

It remains to check that the inequalities

$$1 < \frac{1+a}{1-a} < \frac{(1+2 \cdot a) \cdot (1-a)}{(1-2 \cdot a) \cdot (1+a)}$$

hold if $0 < a < \frac{1}{2}$.

Exercise 12.5. Spell the meaning of terms “perpendicular” and “h-line” and then apply Exercise 10.19.

Exercise 12.8. Apply the main observation (12.7b).

Exercise 12.12. Let X and Y denote the points of intersections of (OP) and Δ'_ρ . Consider an isometry $(OP) \rightarrow \mathbb{R}$ such that O corresponds to 0. Let x, y, p and $\hat{p}$ denote the real numbers corresponding to X, Y, P and $\hat{P}$.

We can assume that $p > 0$ and $x < y$. Note that $\hat{p} = \frac{x+y}{2}$ and

$$\frac{(1+x) \cdot (1-p)}{(1-x) \cdot (1+p)} = \frac{(1+p) \cdot (1-y)}{(1-p) \cdot (1+y)}.$$

It remains to show that all this implies $0 < \hat{p} < p$.

Exercise 12.22. Look at the diagram and think.

Advanced exercise 12.25. By Corollary 10.26 and Theorem 10.6, the right hand sides in the identities survive under an inversion in a circle perpendicular to the absolute.

As usual we assume that the absolute is a unit circle. Let O denotes the h-midpoint of $[PQ]_h$. By the main observation (12.7) we can assume that O is the center of the absolute. In this case O is also the Euclidean midpoint of $[PQ]$.

Set $a = OP = OQ$; in this case we have

$$\begin{aligned} PQ &= 2 \cdot a, & PP' &= QQ' = \frac{1}{a} - a, \\ P'Q' &= 2 \cdot \frac{1}{a}, & PQ' &= QP' = \frac{1}{a} + a. \end{aligned}$$

and

$$PQ_h = \ln \frac{(1+a)^2}{(1-a)^2} = 2 \cdot \ln \frac{1+a}{1-a}.$$

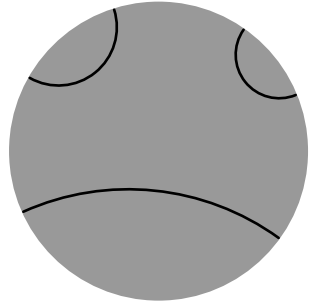
Therefore

$$\begin{aligned} \text{ch}[\tfrac{1}{2} \cdot PQ_h] &= \tfrac{1}{2} \cdot \left(\frac{1+a}{1-a} + \frac{1-a}{1+a} \right) = \sqrt{\frac{PQ' \cdot P'Q}{PP' \cdot QQ'}} = \frac{\frac{1}{a} + a}{\frac{1}{a} - a} = \\ &= \frac{1+a^2}{1-a^2}; & &= \frac{1+a^2}{1-a^2}. \end{aligned}$$

Hence the part (a) follows. Similarly,

$$\begin{aligned} \text{sh}[\tfrac{1}{2} \cdot PQ_h] &= \tfrac{1}{2} \cdot \left(\frac{1+a}{1-a} - \frac{1-a}{1+a} \right) = \sqrt{\frac{PQ \cdot P'Q'}{PP' \cdot QQ'}} = \frac{2}{\frac{1}{a} - a} = \\ &= \frac{2 \cdot a}{1-a^2}; & &= \frac{2 \cdot a}{1-a^2}. \end{aligned}$$

Hence the part (b) follows.



The parts (c) and (d) follow from (a), (b), the definition of hyperbolic tangent and the double-argument identity for hyperbolic cosine, see 12.24.

Exercise 13.3. By triangle inequality, the h-distance from B to $(AC)_h$ is at least 50. It remains to estimate $|\angle_h ABC|$ using Corollary 13.2. The inequalities $\cos \varphi \leq 1 - \frac{1}{10} \cdot \varphi^2$ for $|\varphi| < \frac{\pi}{2}$ and $e^3 > 10$ should help to finish the proof.

Exercise 13.5. Note that the angle of parallelism of B to $(CD)_h$ is bigger than $\frac{\pi}{4}$, and it converges to $\frac{\pi}{4}$ as $CD_h \rightarrow \infty$.

Applying Proposition 13.1, we get that

$$BC_h < \frac{1}{2} \cdot \ln \frac{1 + \frac{1}{\sqrt{2}}}{1 - \frac{1}{\sqrt{2}}} = \ln(1 + \sqrt{2}).$$

The right hand side is the limit of BC_h if $CD_h \rightarrow \infty$. Therefore, $\ln(1 + \sqrt{2})$ is the optimal upper bound.

Exercise 13.6. As usual, we assume that the absolute is a unit circle.

Let PQR be a hyperbolic triangle with a right angle at Q , such that $PQ_h = QR_h$ and the vertices P , Q and R lie on a horocycle.

Without loss of generality, we may assume that Q is the center of the absolute. In this case $\angle_h PQR = \angle PQR = \pm \frac{\pi}{2}$ and $PQ = QR$.

Note that Euclidean circle passing thru P , Q and R is tangent to the absolute. Conclude that $PQ = \frac{1}{\sqrt{2}}$. Apply 12.9 to find PQ_h .

Exercise 13.9. Apply AAA-congruence condition (13.8).

Exercise 13.12. Apply Proposition 13.11. Use that $e > 2$ and in particular the function $r \mapsto e^{-r}$ is decreasing.

Exercise 13.14. Apply the hyperbolic Pythagorean theorem and the definition of hyperbolic cosine.

Exercise 14.1. Assume that a triple of noncollinear points P , Q and R are mapped to one line ℓ . Note that all three lines (PQ) , (QR) and (RP) are mapped to ℓ . Therefore, any line which connects two points on these three lines is mapped to ℓ .

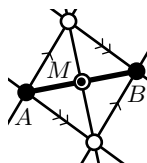
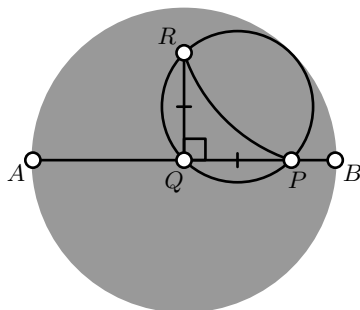
Note that any point in the plane lies on a line passing thru two distinct points on these three lines. Therefore, the whole plane is mapped to ℓ . The latter contradicts that the map is a bijection.

Exercise 14.2. Assume the two distinct lines ℓ and m are mapped to the intersecting lines ℓ' and m' . Let P' denotes their point of intersection.

Let P be the inverse image of P' . By the definition of affine map, it has to lie on both ℓ and m ; that is, ℓ and m are intersecting. Hence the result.

Exercise 14.3. According to the remark before the exercise, it is sufficient to construct the midpoint of $[AB]$ with a ruler and a parallel tool.

Guess a construction from the diagram.



Exercise 14.4. Let O , E , A and B denote the points with the coordinates $(0, 0)$, $(1, 0)$, $(a, 0)$ and $(b, 0)$ correspondingly.

To construct a point W with the coordinates $(0, a + b)$, try to construct two parallelograms $ABPQ$ and $BWPQ$.

To construct Z with coordinates $(0, a \cdot b)$ choose a line $(OE') \neq (OE)$ and try to construct the points $A' \in (OE')$ and $Z \in (OE)$ so that $\triangle OEE' \sim \triangle OAA'$ and $\triangle OE'B \sim \triangle OA'Z$.

Exercise 14.5. Draw two parallel chords $[XX']$ and $[YY']$. Set $Z = (XY) \cap (X'Y')$ and $Z' = (XY') \cap (X'Y)$. Note that (ZZ') passes thru the center.

Repeat the same construction for another pair of parallel chords. The center lies in the intersection of the obtained lines.

Exercise 14.6. Assume a construction produces two perpendicular lines. Apply a shear mapping which changes the angle between the lines.

Note that it transforms the construction to the same construction for other free choices points. Therefore, this construction does not produce perpendicular lines in general (it might be the center only by coincidence).

Exercise 14.11. Apply 10.25 and 14.10.

Exercise 14.12. Fix a line ℓ . Choose a circle Γ with its center not on ℓ . Let Ω be the inverse of ℓ in Γ ; note that Ω is a circle.

Let ι_Γ and ι_Ω denote the inversions in Γ and Ω . Apply 10.26 to show that the composition $\iota_\Gamma \circ \iota_\Omega \circ \iota_\Gamma$ is the reflection in ℓ .

Exercise 15.1. Since O , P and P' lie on one line we have that the coordinates of P' are proportional to the coordinates of P . The y coordinate of P' has to be equal to 1. Therefore, P' has coordinates $(\frac{1}{y}, 1, \frac{x}{y})$.

Exercise 15.5. Assume that (AB) meets $(A'B')$ at O . Since $(AB') \parallel (BA')$, we get that $\triangle OAB' \sim \triangle OBA'$ and

$$\frac{OA}{OB} = \frac{OB'}{OA'}.$$

Similarly, since $(AC') \parallel (CA')$, we get that

$$\frac{OA}{OC} = \frac{OC'}{OA'}.$$

Therefore

$$\frac{OB}{OC} = \frac{OC'}{OB'}.$$

Applying the SAS similarity condition, we get that $\triangle OBC' \sim \triangle OCB'$. Therefore, $(BC') \parallel (CB')$.

The case $(AB) \parallel (A'B')$ is similar.

Exercise 15.6. Assume the contrary. Choose two parallel lines ℓ and m . Let L and M be their dual points. Set $s = (ML)$, then its dual point S has to lie on both ℓ and m — a contradiction.

Exercise 15.8. Assume $M = (a, b)$ and the line s is given by the equation $p \cdot x + q \cdot y = 1$. Then $M \in s$ is equivalent to $p \cdot a + q \cdot b = 1$.

The latter is equivalent to $m \ni S$ where m is the line given by the equation $a \cdot x + b \cdot y = 1$ and $S = (p, q)$.

To extend this bijection to the whole projective plane, assume that (1) the ideal line corresponds to the origin and (2) the ideal point given by the pencil of the lines $b \cdot x - a \cdot y = c$ for different values of c corresponds to the line given by the equation $a \cdot x + b \cdot y = 0$.

Exercise 15.10. Assume one set of concurrent lines a, b, c , and another set of concurrent lines a', b', c' are given. Set

$$\begin{aligned} P &= b \cap c', & Q &= c \cap a', & R &= a \cap b', \\ P' &= b' \cap c, & Q' &= c' \cap a, & R' &= a' \cap b. \end{aligned}$$

Then the lines (PP') , (QQ') and (RR') are concurrent.

Exercise 15.11. To solve (a), assume (AA') and (BB') are the given lines and C is the given point. Apply the dual Desargues' theorem (15.9) to construct C' so that (AA') , (BB') and (CC') are concurrent. Since $(AA') \parallel (BB')$, we get that $(AA') \parallel (BB') \parallel (CC')$.

A similar solution can be build on the dual Pappus' theorem, see Exercise 15.10.

For part (b), apply one of the discussed theorems to construct a line parallel to the given one and then apply part (a).

Exercise 15.12. Let A, B, C and D be the point provided by Axiom p-III. Given a line ℓ , we can assume that $A \notin \ell$, otherwise permute the labels of the points. Then by axioms p-I and p-II, the three lines (AB) , (AC) and (AD) intersect ℓ at distinct points. In particular, ℓ contains at least three points.

Exercise 15.13. Let A, B, C and D be the point provided by Axiom p-III. Show that the lines (AB) , (BC) , (CD) and (DA) satisfy Axiom p-III'. The proof of the converse is similar.

Exercise 15.14. Let ℓ be a line with $n + 1$ points on it.

By Axiom p-III, given any line m there is a point P which does not lie on ℓ nor on m .

By axioms p-I and p-II, there is a bijection between the lines passing thru P and the points on ℓ . In particular, there are exactly $n + 1$ lines passing thru P .

The same way there is bijection between the lines passing thru P and the points on m . Hence (a) follows.

Fix a point X . By Axiom p-I, any point Y in the plane lies in a unique line passing thru X . From part (a), each such line contains X and yet n point. Hence (b) follows.

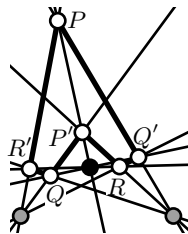
To solve (c), show that the equation $n^2 + n + 1 = 10$ does not admit an integer solution and then apply part (b).

To solve (d), count the number of lines crossing a given line using the part (a) and apply (b).

Exercise 16.2. Applying the Pythagorean theorem, we get that

$$\cos AB_s = \cos AC_s \cdot \cos BC_s = \frac{1}{2}.$$

Therefore, $AB_s = \frac{\pi}{3}$.



Alternatively, look at the tessellation of the sphere on the picture; it is made from 24 copies of $\triangle_s ABC$ and yet 8 equilateral triangles. From the symmetry of this tessellation, it follows that $[AB]_s$ occupies $\frac{1}{6}$ of the equator.

Exercise 16.6. Consider the inversion of the base in a sphere with the center at the tip of the cone and apply Theorem 16.3.

Exercise 16.7. Note that points on Ω do not move. Moreover, the points inside Ω are mapped outside of Ω and the other way around.

Further, note that this map sends circles to circles; moreover, the perpendicular circles are mapped to perpendicular circles. In particular, the circles perpendicular to Ω are mapped to themselves.

Consider arbitrary point $P \notin \Omega$. Let P' denotes the inverse of P in Ω . Choose two distinct circles which pass thru P and P' . According to Corollary 10.17, $\Gamma_1 \perp \Omega$ and $\Gamma_2 \perp \Omega$.

Therefore, the inversion in Ω sends Γ_1 to itself and the same holds for Γ_2 .

The image of P has to lie on Γ_1 and Γ_2 . Since the image of P is distinct from P , we get that it has to be P' .

Exercise 16.8. Apply Theorem 16.3b.

Exercise 16.9. Set $z = P'Q'$. Note that $\frac{y}{z} \rightarrow 1$ as $x \rightarrow 0$.

It remains to show that

$$\lim_{x \rightarrow 0} \frac{z}{x} = \frac{2}{1 + OP^2}.$$

Recall that the stereographic projection is the inversion in the sphere Υ with the center at the south pole S restricted to the plane Π . Show that there is a plane Λ passing thru S, P, Q, P' and Q' . In the plane Λ , the map $Q \mapsto Q'$ is an inversion in the circle $\Upsilon \cap \Lambda$.

This reduces the problem to Euclidean plane geometry. The remaining calculations in Λ are similar to those in the proof of Lemma 13.10.

Exercise 16.10. (a). Observe and use that $OA' = OB' = OC'$.

(b). Note that the medians of spherical triangle ABC map to the medians of Euclidean triangle $A'B'C'$. It remains to apply Theorem 8.5 for $\triangle A'B'C'$.

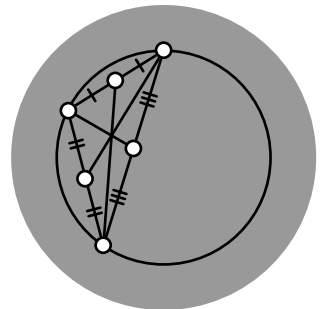
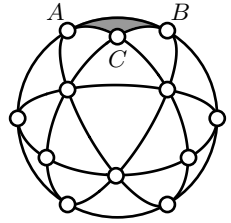
Exercise 17.1. Let N, O, S, P, P' and $\hat{P}$ be as on the diagram on page 134.

Notice that $\triangle NOP \sim \triangle NP'S \sim \triangle P'\hat{P}P$ and $2 \cdot NO = NS$. It remains to do some algebraic manipulations.

Exercise 17.3. Consider the bijection $P \leftrightarrow \hat{P}$ of the h-plane with absolute Ω . Note that $\hat{P} \in [A_i B_i]$ if and only if $P \in \Gamma_i$.

Exercise 17.4. The observation follows since the reflection in the perpendicular bisector of $[PQ]$ is a motion of the Euclidean plane, and a motion of the h-plane as well.

Without loss of generality, we may assume that the center of the circumcircle coincides with the center of the



absolute. In this case the h-medians of the triangle coincide with the Euclidean medians. It remains to apply Theorem 8.5.

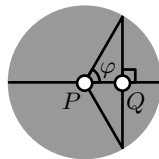
Exercise 17.5. Let $\hat{\ell}$ and $\hat{m}$ denote the h-lines in the conformal model which correspond to ℓ and m . We need to show that $\hat{\ell} \perp \hat{m}$ as arcs in the Euclidean plane.

The point Z , where s meets t , is the center of the circle Γ containing $\hat{\ell}$.

If $\bar{m}$ is passing thru Z , then the inversion in Γ exchanges the ideal points of $\hat{\ell}$. In particular, $\hat{\ell}$ maps to itself. Hence the result.

Exercise 17.6. Let Q be the foot point of P on the line and φ be the angle of parallelism. We can assume that P is the center of the absolute. Therefore $PQ = \cos \varphi$ and

$$PQ_h = \frac{1}{2} \cdot \ln \frac{1 + \cos \varphi}{1 - \cos \varphi}.$$



Exercise 17.7. Apply Exercise 17.6 for $\varphi = \frac{\pi}{3}$.

Exercise 17.8. Note that

$$b = \frac{1}{2} \cdot \ln \frac{1+t}{1-t};$$

therefore

$$\textbf{1} \quad \text{ch } b = \frac{1}{2} \cdot \left(\sqrt{\frac{1+t}{1-t}} + \sqrt{\frac{1-t}{1+t}} \right) = \frac{1}{\sqrt{1-t^2}}.$$

The same way we get that

$$\textbf{2} \quad \text{ch } c = \frac{1}{\sqrt{1-u^2}}.$$

Let X and Y are the ideal points of $(BC)_h$. Applying the Pythagorean theorem (6.4) again, we get that

$$CX = CY = \sqrt{1-t^2}.$$

Therefore,

$$a = \frac{1}{2} \cdot \ln \frac{\sqrt{1-t^2} + s}{\sqrt{1-t^2} - s},$$

and

$$\begin{aligned} \textbf{3} \quad \text{ch } a &= \frac{1}{2} \cdot \left(\sqrt{\frac{\sqrt{1-t^2} + s}{\sqrt{1-t^2} - s}} + \sqrt{\frac{\sqrt{1-t^2} - s}{\sqrt{1-t^2} + s}} \right) = \\ &= \frac{\sqrt{1-t^2}}{\sqrt{1-t^2} - s^2} = \\ &= \frac{\sqrt{1-t^2}}{\sqrt{1-u^2}}. \end{aligned}$$

Finally, note that **1**, **2** and **3** imply the theorem.

Exercise 17.10. In the Euclidean plane, the circle Γ_2 is tangent to k ; that is, the point T of intersection of Γ_2 and k is unique. It defines a unique line (PT) parallel to ℓ .

Exercise 18.1. Use that $|z|^2 = z \cdot \bar{z}$ for $z = v, w$ and $v \cdot w$.

Exercise 18.2. Given a quadrilateral $ABCD$, we can choose the complex coordinates so that A has complex coordinate 0. Rewrite the terms in the Ptolemy's inequality in terms of the complex coordinates u, v and w of B, C and D ; apply the identity and the triangle inequality.

Exercise 18.3. Let z, v and w denote the complex coordinates of Z, V and W correspondingly. Then

$$\begin{aligned}\angle ZVW + \angle VWZ + \angle WZV &\equiv \arg \frac{w-v}{z-v} + \arg \frac{z-w}{v-w} + \arg \frac{v-z}{w-z} \equiv \\ &\equiv \arg \frac{(w-v) \cdot (z-w) \cdot (v-z)}{(z-v) \cdot (v-w) \cdot (w-z)} \equiv \\ &\equiv \arg(-1) \equiv \\ &\equiv \pi.\end{aligned}$$

Exercise 18.4. Note and use that

$$\angle EOZ = \angle WOZ = \arg v, \quad \frac{OW}{OZ} = \frac{OZ}{OW} = |v|.$$

Exercise 18.6. Note that

$$\begin{aligned}\arg \frac{(v-u) \cdot (z-w)}{(v-w) \cdot (z-u)} &\equiv \arg \frac{v-u}{z-u} + \arg \frac{z-w}{v-w} = \\ &= \angle ZUV + \angle VWZ.\end{aligned}$$

The statement follows since the value $\frac{(v-u) \cdot (z-w)}{(v-w) \cdot (z-u)}$ is real if and only if

$$2 \cdot \arg \frac{(v-u) \cdot (z-w)}{(v-w) \cdot (z-u)} \equiv 0.$$

Exercise 18.7. Check the following identity:

$$\begin{aligned}\frac{(v-u) \cdot (z-w)}{(v-w) \cdot (z-u)} \cdot \frac{(v'-u') \cdot (z'-w')}{(v'-w') \cdot (z'-u')} &= \frac{(v-u) \cdot (u'-v')}{(v-v') \cdot (u'-u)} \cdot \frac{(z-w) \cdot (w'-z')}{(z-z') \cdot (w'-w)} \\ &\cdot \frac{(v-v') \cdot (w'-w)}{(v-w) \cdot (w'-v')} \cdot \frac{(z-z') \cdot (u'-u)}{(z-u) \cdot (u'-z')}.\end{aligned}$$

By Theorem 18.5, five from six cross ratios in the this identity are real. Therefore so is the sixth cross ratio; it remains to apply the theorem again.

Exercise 18.9. We can choose the complex coordinates so that the points O, E, A, B and C have coordinates 0, 1, $1+i$, $2+i$ and $3+i$ correspondingly.

Set $\angle EOA = \alpha$, $\angle EOB = \beta$ and $\angle EOC = \gamma$. Note that

$$\begin{aligned}\alpha + \beta + \gamma &\equiv \arg(1+i) + \arg(2+i) + \arg(3+i) = \\ &\equiv \arg[(1+i) \cdot (2+i) \cdot (3+i)] = \\ &\equiv \arg[10 \cdot i] = \\ &\equiv \frac{\pi}{2}.\end{aligned}$$

Note that these three angles are acute and conclude that $\alpha + \beta + \gamma = \frac{\pi}{2}$.

Exercise 18.13. Show that the inverse of each elementary transformation is elementary and use Proposition 18.11.

Exercise 18.14. The fractional linear transformation

$$f(z) = \frac{(z_1 - z_\infty) \cdot (z - z_0)}{(z_1 - z_0) \cdot (z - z_\infty)}$$

meets the conditions.

To show the uniqueness, assume there is another fractional linear transformation $g(z)$ which meets the conditions. Then the composition $h = g \circ f^{-1}$ is a fractional linear transformation; set $h(z) = \frac{a \cdot z + b}{c \cdot z + d}$.

Note that $h(\infty) = \infty$; therefore, $c = 0$. Further, $h(0) = 0$ implies $b = 0$. Finally, since $h(1) = 1$, we get that $\frac{a}{d} = 1$. Therefore, h is the *identity*; that is, $h(z) = z$ for any z . It follows that $g = f$.

Exercise 18.15. Let Z' be the inverse of the point Z . Assume that the circle of the inversion has center W and radius r . Let z , z' and w denote the complex coordinate of the points Z , Z' and W correspondingly.

By the definition of inversion:

$$\arg(z - w) = \arg(z' - w), \quad |z - w| \cdot |z' - w| = r^2$$

It follows that $(\bar{z}' - \bar{w}) \cdot (z - w) = r^2$. Equivalently,

$$z' = \left(\frac{\bar{w} \cdot z + [r^2 - |w|^2]}{z - w} \right).$$

Exercise 18.17. Check the statement for each elementary transformation. Then apply Proposition 18.11.

Exercise 18.19. Note that $f = \frac{a \cdot z + b}{c \cdot z + d}$ preserves the unit circle $|z| = 1$. Use Corollary 10.26 and Proposition 18.11 to show that f commutes with the inversion $z \mapsto 1/\bar{z}$. In other words, $1/\overline{f(z)} = f(1/\bar{z})$ or

$$\frac{\bar{c} \cdot \bar{z} + \bar{d}}{\bar{a} \cdot \bar{z} + \bar{b}} = \frac{a/\bar{z} + b}{c/\bar{z} + d}$$

for any $z \in \hat{\mathbb{C}}$. The latter identity leads to the required statement. The condition $|w| < |v|$ follows since $f(0) \in \mathbb{D}$.

Exercise 18.20. Note that the inverses of the points z and w have complex coordinates $1/\bar{z}$ and $1/\bar{w}$. Apply Exercise 12.25 and simplify.

The second part follows since the function $x \mapsto \operatorname{th}(\frac{1}{2} \cdot x)$ is increasing.

Exercise 18.21. Apply Schwarz–Pick theorem for a function f such that $f(0) = 0$ and then apply Lemma 12.9.

Exercise 19.3. Let O be the point of intersection of the lines. Draw a line ℓ thru the given point P and O . Construct a circle Γ , tangent to both lines, which crosses ℓ . Let X denotes one of the points of intersections.

Consider the homothety with the center at O which sends X to P . The image of Γ is the needed circle.

Exercise 19.6, (a). Show that the angles marked the same way on the on the diagram of regular pentagon are equal to each other. Conclude that $XC = BC$ and $\triangle ABC \sim \triangle AXB$. Therefore

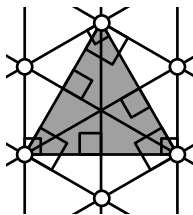
$$\frac{AB}{AC} = \frac{AX}{AB} = \frac{AC - AB}{AB} = \frac{AC}{AB} - 1.$$

It remains to solve for $\frac{AC}{AB}$.

(b). Choose two points P and Q and use the compass-and-ruler calculator to construct two points V and W such that $VW = \frac{1+\sqrt{5}}{2} \cdot PQ$. Then construct a pentagon with the sides PQ and diagonals VW .

Exercise 19.7. Note that with a set square we can construct a line parallel to given line thru the given point. It remains to modify the construction in Exercise 14.3.

Exercise 19.9. Assume that two vertices have rational coordinates, say (a_1, b_1) and (a_2, b_2) . Find the coordinates of the third vertex. Use that the number $\sqrt{3}$ is irrational to show that the third vertex is an irrational point.



Exercise 19.10. Guess the construction from the diagram. Prove that it verifies that the triangle is equilateral.

Exercise 19.11. Apply the same argument as in Exercise 10.9.

Exercise 19.12. Consider the perspective projection as in Exercise 15.1. Let $A = (1, 1, 1)$, $B = (1, 3, 1)$ and $M = (1, 2, 1)$. Note that M is the midpoint of $[AB]$.

Their images are $A' = (1, 1, 1)$, $B' = (\frac{1}{3}, 1, \frac{1}{3})$ and $M' = (\frac{1}{2}, 1, \frac{1}{2})$. Clearly, M' is not the midpoint of $[A'B']$.

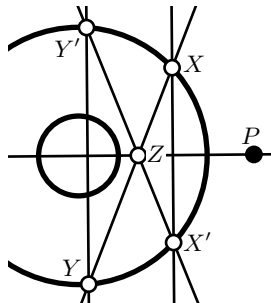
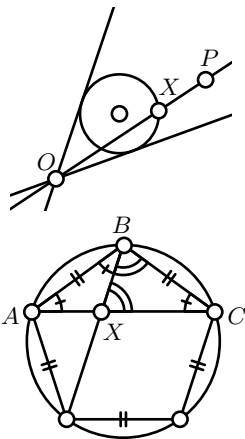
Exercise 19.15. The line v polar to V is tangent to Γ . Since $V \in p$, by Claim 19.14, we get that $P \in v$; that is, $(PV) = v$. Hence the statement follows.

Exercise 19.16. Choose a point P outside of the bigger circle. Construct the lines dual to P for both circles. Note that these two lines are parallel.

Assume that the lines intersect the bigger circle at two pairs of points X, X' and Y, Y' . Set $Z = (XY) \cap (X'Y')$. Note that the line (PZ) passes thru the common center.

The center is the intersection of (PZ) and another line constructed the same way.

Exercise 20.1. Assume the contrary; that is, there is a point $W \in [XY]$ such that $W \notin \triangle ABC$.



Without loss of generality, we may assume that W and A lie on the opposite sides of the line (BC) .

It implies that both segments $[WX]$ and $[WY]$ intersect (BC) . By Axiom II, $W \in (BC)$ — a contradiction.

Exercise 20.3. To prove the “only if” part, consider the line passing thru the vertex which is parallel to the opposite side.

To prove the “if” part, use Pasch’s theorem (3.12).

Exercise 20.4. Assume the contrary; that is, a solid square $\mathcal{Q}$ can be presented as a union of a finite collection of segments $[A_1B_1], \dots, [A_nB_n]$ and one-point sets $\{C_1\}, \dots, \{C_k\}$.

Note that $\mathcal{Q}$ contains an infinite number of mutually nonparallel segments. Therefore, we can choose a segment $[PQ]$ in $\mathcal{Q}$ which is not parallel to any of the segments $[A_1B_1], \dots, [A_nB_n]$.

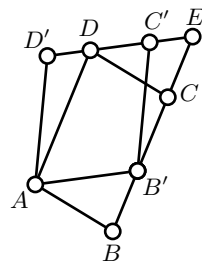
It follows that $[PQ]$ has at most one common point with each of the sets $[A_iB_i]$ and $\{C_i\}$. Since $[PQ]$ contains infinite number of points, we arrive to a contradiction.

Exercise 20.5. First note that among elementary sets only one-point sets can be subsets of the a circle. It remains to note that any circle contains an infinite number of points.

Exercise 20.15. Let E denotes the point of intersection of the lines (BC) and $(C'D')$.

Use Proposition 20.14 to prove the following two identities:

$$\begin{aligned} \text{area}(\blacksquare AB'ED) &= \text{area}(\blacksquare ABCD), \\ \text{area}(\blacksquare AB'ED) &= \text{area}(\blacksquare AB'C'D'). \end{aligned}$$



Hence the statement follows.

Exercise 20.17. Without loss of generality, we may assume that the angles ABC and BCA are acute.

Let A' and B' denote the foot points of A and B on (BC) and (AC) correspondingly. Note that $h_A = AA'$ and $h_B = BB'$.

Note that $\triangle AA'C \sim \triangle BB'C$; indeed the angle at C is shared and the angles at A' and B' are right. In particular

$$\frac{AA'}{BB'} = \frac{AC}{BC}$$

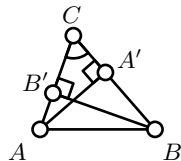
or, equivalently,

$$h_A \cdot BC = h_B \cdot AC.$$

Along the same lines, we get that

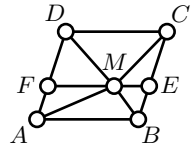
$$h_C \cdot AB = h_B \cdot AC.$$

Hence the statement follows.



Exercise 20.18. Draw the line ℓ thru M parallel to $[AB]$ and $[CD]$; it subdivides $\blacksquare ABCD$ into two solid parallelograms which will be denoted by $\blacksquare ABEF$ and $\blacksquare CDFE$. In particular,

$$\text{area}(\blacksquare ABCD) = \text{area}(\blacksquare ABEF) + \text{area}(\blacksquare CDFE).$$



By Proposition 20.14 and Theorem 20.16 we get that

$$\text{area}(\blacktriangle ABM) = \frac{1}{2} \cdot \text{area}(\blacksquare ABEF),$$

$$\text{area}(\blacktriangle CDM) = \frac{1}{2} \cdot \text{area}(\blacksquare CDFE)$$

and hence the result.

Exercise 20.19. Let h_A and h_C denote the distances from A and C to the line (BD) correspondingly. According to Theorem 20.16,

$$\text{area}(\blacktriangle ABM) = \frac{1}{2} \cdot h_A \cdot BM; \quad \text{area}(\blacktriangle BCM) = \frac{1}{2} \cdot h_C \cdot BM;$$

$$\text{area}(\blacktriangle CDM) = \frac{1}{2} \cdot h_C \cdot DM; \quad \text{area}(\blacktriangle ADM) = \frac{1}{2} \cdot h_A \cdot DM.$$

Therefore

$$\begin{aligned} \text{area}(\blacktriangle ABM) \cdot \text{area}(\blacktriangle CDM) &= \frac{1}{4} \cdot h_A \cdot h_C \cdot DM \cdot BM = \\ &= \text{area}(\blacktriangle BCM) \cdot \text{area}(\blacktriangle DAM). \end{aligned}$$

Exercise 20.20. Let I be the incenter of $\triangle ABC$. Note that $\blacktriangle ABC$ can be subdivided into $\blacktriangle IAB$, $\blacktriangle IBC$ and $\blacktriangle ICA$.

It remains to apply Theorem 20.16 to each of these triangles and sum up the results.

Exercise 20.22. Assuming $a > b$, we subdivided $\mathcal{Q}_c$ into $\mathcal{Q}_{a-b}$ and four triangles congruent to $\mathcal{T}$. Therefore

$$\textcircled{1} \quad \text{area } \mathcal{Q}_c = \text{area } \mathcal{Q}_{a-b} + 4 \cdot \text{area } \mathcal{T}.$$

According to Theorem 20.16, $\text{area } \mathcal{T} = \frac{1}{2} \cdot a \cdot b$. Therefore, the identity $\textcircled{1}$ can be written as

$$c^2 = (a - b)^2 + 2 \cdot a \cdot b.$$

Simplifying, we get the Pythagorean theorem.

The case $a = b$ is yet simpler. The case $b > a$ can be done the same way.

Exercise 20.23. If X is a point inside of $\triangle ABC$, then $\blacktriangle ABC$ is subdivided into $\blacktriangle ABX$, $\blacktriangle BCX$ and $\blacktriangle CAX$. Therefore

$$\text{area}(\blacktriangle ABX) + \text{area}(\blacktriangle BCX) + \text{area}(\blacktriangle CAX) = \text{area}(\blacktriangle ABC).$$

Set $a = AB = BC = CA$. Let h_1 , h_2 and h_3 denote the distances from X to the sides $[AB]$, $[BC]$ and $[CA]$. Then by Theorem 20.16,

$$\text{area}(\blacktriangle ABX) = \frac{1}{2} \cdot h_1 \cdot a,$$

$$\text{area}(\blacktriangle BCX) = \frac{1}{2} \cdot h_2 \cdot a,$$

$$\text{area}(\blacktriangle CAX) = \frac{1}{2} \cdot h_3 \cdot a.$$

Therefore,

$$h_1 + h_2 + h_3 = \frac{2}{a} \cdot \text{area}(\triangle ABC).$$

Exercise 20.25. Let X be a point inside $\triangle ABC$. Let Y denotes the point of intersection of (AX) and $[BC]$.

Let b and c denote the distances from B and C to the line (AX) .

By Theorem 20.16, we get the following equivalences:

$$\text{area}(\triangle AXB) = \text{area}(\triangle AXC),$$

$$\Updownarrow$$

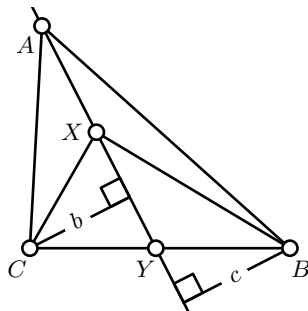
$$b = c,$$

$$\Updownarrow$$

$$\text{area}(\triangle AYB) = \text{area}(\triangle AYC),$$

$$\Updownarrow$$

$$BY = CY.$$



Exercise 20.26. Let M denotes the intersection of two medians $[AA']$ and $[BB']$. From Exercise 20.25 we have $\text{area}(\triangle ABM) = \text{area}(\triangle ACM)$ and $\text{area}(\triangle ABM) = \text{area}(\triangle CBM)$. Therefore, $\text{area}(\triangle BCM) = \text{area}(\triangle ACM)$.

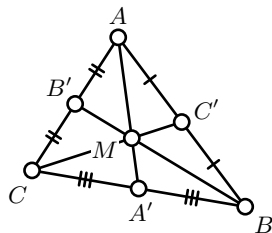
According to Exercise 20.25, M lies on the median $[CC']$. That is, medians $[AA']$, $[BB']$ and $[CC']$ intersect at one point M .

By Theorem 20.16, we get that

$$\begin{aligned} \text{area}(\triangle C'AM) &= \frac{1}{2} \cdot \text{area}(\triangle BAM) \\ &= \frac{1}{2} \cdot \text{area}(\triangle CAM) \end{aligned}$$

Applying Claim 20.24, we get that

$$\frac{MC'}{MC} = \frac{\text{area}(\triangle C'AM)}{\text{area}(\triangle CAM)} = \frac{1}{2}.$$



Exercise 20.27. Let $\mathcal{P}_n$ and $\mathcal{Q}_n$ be the solid regular n -gons so that Γ is inscribed in $\mathcal{Q}_n$ and circumscribed around $\mathcal{P}_n$. Clearly,

$$\mathcal{P}_n \subset \mathcal{D} \subset \mathcal{Q}_n.$$

Show that $\frac{\text{area } \mathcal{P}_n}{\text{area } \mathcal{Q}_n} = (\cos \frac{\pi}{n})^2$; in particular,

$$\frac{\text{area } \mathcal{P}_n}{\text{area } \mathcal{Q}_n} \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

Next show that $\text{area } \mathcal{Q}_n < 100$, say for all $n \geq 100$.

These two statements imply that $(\text{area } \mathcal{Q}_n - \text{area } \mathcal{P}_n) \rightarrow 0$. Hence the result.

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