

Normal (geometry)

In geometry, a **normal** is an object such as a line, ray, or vector that is perpendicular to a given object. For example, in two dimensions, the **normal line** to a curve at a given point is the line perpendicular to the tangent line to the curve at the point. A normal vector may have length one (a unit vector) or its length may represent the curvature of the object (a curvature vector); its algebraic sign may indicate sides (interior or exterior).

In three dimensions, a **surface normal**, or simply **normal**, to a surface at point *P* is a vector perpendicular to the tangent plane of the surface at *P*. The word "normal" is also used as an adjective: a line normal to a plane, the *normal* component of a force, the **normal vector**, etc. The concept of normality generalizes to orthogonality (right angles).

The concept has been generalized to differentiable manifolds of arbitrary dimension embedded in a Euclidean space. The **normal vector space** or **normal space** of a manifold at point *P* is the set of vectors which are orthogonal to the tangent space at *P*. Normal vectors are of special interest in the case of smooth curves and smooth surfaces.

The normal is often used in 3D computer graphics (notice the singular, as only one normal will be defined) to determine a surface's orientation toward a light source for flat shading, or the orientation of each of the surface's corners (vertices) to mimic a curved surface with Phong shading.

A polygon and its two normal vectors

A normal to a surface at a point is the same as a normal to the tangent plane to the surface at the same point.

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Normal to surfaces in 3D space

Calculating a surface normal

For a convex polygon (such as a triangle), a surface normal can be calculated as the vector cross product of two (non-parallel) edges of the polygon.

For a plane given by the equation $\mathbf{ax} + \mathbf{by} + \mathbf{cz} + \mathbf{d} = 0$, the vector $\mathbf{n} = (\mathbf{a}, \mathbf{b}, \mathbf{c})$ is a normal.

A curved surface showing the unit normal vectors (blue arrows) to the surface

For a plane whose equation is given in parametric form

$$\mathbf{r}(s, t) = \mathbf{r}_0 + s\mathbf{p} + t\mathbf{q},$$

where \mathbf{r}_0 is a point on the plane and \mathbf{p}, \mathbf{q} are non-parallel vectors pointing along the plane, a normal to the plane is a vector normal to both \mathbf{p} and \mathbf{q} , which can be found as the cross product $\mathbf{n} = \mathbf{p} \times \mathbf{q}$.

If a (possibly non-flat) surface S in 3-space \mathbf{R}^3 is parameterized by a system of curvilinear coordinates $\mathbf{r}(s, t) = (x(s,t), y(s,t), z(s,t))$, with s and t real variables, then a normal to S is by definition a normal to a tangent plane, given by the cross product of the partial derivatives

$$\mathbf{n} = \frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t}.$$

If a surface S is given implicitly as the set of points (x, y, z) satisfying $F(x, y, z) = 0$, then a normal at a point (x, y, z) on the surface is given by the gradient

$$\mathbf{n} = \nabla F(x, y, z).$$

since the gradient at any point is perpendicular to the level set S .

For a surface S in \mathbf{R}^3 given as the graph of a function $z = f(x, y)$, an upward-pointing normal can be found either from the parametrization $\mathbf{r}(x, y) = (x, y, f(x, y))$, giving:

$$\mathbf{n} = \frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} = \left(1, 0, \frac{\partial f}{\partial x}\right) \times \left(0, 1, \frac{\partial f}{\partial y}\right) = \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right);$$

or more simply from its implicit form $F(x, y, z) = z - f(x, y) = 0$, giving $\mathbf{n} = \nabla F(x, y, z) = \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right)$.

Since a surface does not have a tangent plane at a singular point, it has no well-defined normal at that point: for example, the vertex of a cone. In general, it is possible to define a normal almost everywhere for a surface that is Lipschitz continuous.

Choice of normal

The normal to a (hyper)surface is usually scaled to have unit length, but it does not have a unique direction, since its opposite is also a unit normal. For a surface which is the topological boundary of a set in three dimensions, one can distinguish between the **inward-pointing normal** and **outer-pointing normal**. For an oriented surface, the normal is usually determined by the right-hand rule or its analog in higher dimensions.

If the normal is constructed as the cross product of tangent vectors (as described in the text above), it is a pseudovector.

A vector field of normals to a surface

Transforming normals

Note: in this section we only use the upper 3x3 matrix, as translation is irrelevant to the calculation

When applying a transform to a surface it is often useful to derive normals for the resulting surface from the original normals.

Specifically, given a 3x3 transformation matrix \mathbf{M} , we can determine the matrix \mathbf{W} that transforms a vector \mathbf{n} perpendicular to the tangent plane \mathbf{t} into a vector \mathbf{n}' perpendicular to the transformed tangent plane $\mathbf{M}\mathbf{t}$, by the following logic:

Write \mathbf{n}' as $\mathbf{W}\mathbf{n}$. We must find \mathbf{W} .

$\mathbf{W}\mathbf{n}$ perpendicular to $\mathbf{M}\mathbf{t}$

$$\begin{aligned} \iff (\mathbf{W}\mathbf{n}) \cdot (\mathbf{M}\mathbf{t}) &= 0 \\ \iff (\mathbf{W}\mathbf{n})^T (\mathbf{M}\mathbf{t}) &= 0 \\ \iff (\mathbf{n}^T \mathbf{W}^T) (\mathbf{M}\mathbf{t}) &= 0 \\ \iff \mathbf{n}^T (\mathbf{W}^T \mathbf{M}) \mathbf{t} &= 0 \end{aligned}$$

Clearly choosing \mathbf{W} such that $\mathbf{W}^T \mathbf{M} = \mathbf{I}$, or $\mathbf{W} = (\mathbf{M}^{-1})^T$, will satisfy the above equation, giving a $\mathbf{W}\mathbf{n}$ perpendicular to $\mathbf{M}\mathbf{t}$, or an \mathbf{n}' perpendicular to \mathbf{t}' , as required.

Therefore, one should use the inverse transpose of the linear transformation when transforming surface normals. The inverse transpose is equal to the original matrix if the matrix is orthonormal, i.e. purely rotational with no scaling or shearing.

Hypersurfaces in n -dimensional space

For an $(n-1)$ -dimensional hyperplane in n -dimensional space \mathbf{R}^n given by its parametric representation

$$\mathbf{r}(t_1, \dots, t_{n-1}) = \mathbf{p}_0 + t_1 \mathbf{p}_1 + \dots + t_{n-1} \mathbf{p}_{n-1},$$

where \mathbf{p}_0 is a point on the hyperplane and \mathbf{p}_i for $i = 1, \dots, n-1$ are linearly independent vectors pointing along the hyperplane, a normal to the hyperplane is any vector \mathbf{n} in the null space of the matrix $\mathbf{P} = [\mathbf{p}_1 \dots \mathbf{p}_{n-1}]$, meaning $\mathbf{P}\mathbf{n} = \mathbf{0}$. That is, any vector orthogonal to all in-plane vectors is by definition

a surface normal. Alternatively, if the hyperplane is defined as the solution set of a single linear equation $a_1x_1 + \dots + a_nx_n = c$, then the vector $\mathbf{n} = (a_1, \dots, a_n)$ is a normal.

The definition of a normal to a surface in three-dimensional space can be extended to $(n-1)$ -dimensional hypersurfaces in \mathbf{R}^n . A hypersurface may be locally defined implicitly as the set of points (x_1, x_2, \dots, x_n) satisfying an equation $F(x_1, x_2, \dots, x_n) = 0$, where F is a given scalar function. If F is continuously differentiable then the hypersurface is a differentiable manifold in the neighbourhood of the points where the gradient is not zero. At these points a normal vector is given by the gradient:

$$\mathbf{n} = \nabla F(x_1, x_2, \dots, x_n) = \left(\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}, \dots, \frac{\partial F}{\partial x_n} \right).$$

The **normal line** is the one-dimensional subspace with basis $\{\mathbf{n}\}$.

Varieties defined by implicit equations in n -dimensional space

A differential variety defined by implicit equations in the n -dimensional space \mathbf{R}^n is the set of the common zeros of a finite set of differentiable functions in n variables

$$f_1(x_1, \dots, x_n), \dots, f_k(x_1, \dots, x_n).$$

The Jacobian matrix of the variety is the $k \times n$ matrix whose i -th row is the gradient of f_i . By the implicit function theorem, the variety is a manifold in the neighborhood of a point where the Jacobian matrix has rank k . At such a point P , the **normal vector space** is the vector space generated by the values at P of the gradient vectors of the f_i .

In other words, a variety is defined as the intersection of k hypersurfaces, and the normal vector space at a point is the vector space generated by the normal vectors of the hypersurfaces at the point.

The **normal (affine) space** at a point P of the variety is the affine subspace passing through P and generated by the normal vector space at P .

These definitions may be extended *verbatim* to the points where the variety is not a manifold.

Example

Let V be the variety defined in the 3-dimensional space by the equations

$$xy = 0, \quad z = 0.$$

This variety is the union of the x -axis and the y -axis.

At a point $(a, 0, 0)$, where $a \neq 0$, the rows of the Jacobian matrix are $(0, 0, 1)$ and $(0, a, 0)$. Thus the normal affine space is the plane of equation $x = a$. Similarly, if $b \neq 0$, the normal plane at $(0, b, 0)$ is the plane of equation $y = b$.

At the point $(0, 0, 0)$ the rows of the Jacobian matrix are $(0, 0, 1)$ and $(0, 0, 0)$. Thus the normal vector space and the normal affine space have dimension 1 and the normal affine space is the z -axis.

Uses

- Surface normals are useful in defining surface integrals of vector fields.

- Surface normals are commonly used in 3D computer graphics for lighting calculations (see Lambert's cosine law), often adjusted by normal mapping.
- Render layers containing surface normal information may be used in Digital compositing to change the apparent lighting of rendered elements.
- In computer vision, the shapes of 3D objects are estimated from surface normals using photometric stereo.^[1]

Normal in geometric optics

The **normal ray** is the outward-pointing ray perpendicular to the surface of an optical medium at a given point.^[2] In reflection of light, the angle of incidence and the angle of reflection are respectively the angle between the normal and the incident ray (on the plane of incidence) and the angle between the normal and the reflected ray.

See also

- Ellipsoid normal vector
- Pseudovector
- Dual space
- Vertex normal

Diagram of specular reflection

References

1. Ying Wu. "Radiometry, BRDF and Photometric Stereo" (<http://users.eecs.northwestern.edu/~yingwu/teaching/EECS432/Notes/lighting.pdf>) (PDF). Northwestern University.
2. "The Law of Reflection" (<https://www.physicsclassroom.com/class/refln/Lesson-1/The-Law-of-Reflection>). *The Physics Classroom Tutorial*. Archived (<https://web.archive.org/web/20090427131454/http://www.glenbrook.k12.il.us/GBSSCI/PHYS/CLASS/refln/u13l1c.html>) from the original on April 27, 2009. Retrieved 2008-03-31.

External links

- Weisstein, Eric W. "Normal Vector" (<https://mathworld.wolfram.com/NormalVector.html>). *MathWorld*.
- An explanation of normal vectors ([http://msdn.microsoft.com/en-us/library/bb324491\(VS.85\).aspx](http://msdn.microsoft.com/en-us/library/bb324491(VS.85).aspx)) from Microsoft's MSDN
- Clear pseudocode for calculating a surface normal (http://www.opengl.org/wiki/Calculating_a_Surface_Normal) from either a triangle or polygon.

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